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Persistent Liquidity Effects and Long-Run Money Demand†

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We present a monetary model with segmented asset markets that implies a persistent fall in interest rates after a once-and-for-all increase in liquidity. The gradual propagation mechanism produced by our model is novel in the literature. We provide an analytical characterization of this mechanism, showing that the magnitude of the liquidity effect on impact, and its persistence, depend on the ratio of two parameters: the long-run interest rate elasticity of money demand and the intertemporal substitution elasticity. The model simultaneously explains the short-run "instability" of money demand estimates as well as the stability of long-run interest-elastic money demand. (*JEL* E13, E31, E41, E43, E52, E62)

This paper unifies two main views, or theories, on money demand. One is the transactions-based money demand that emerges in the models of, e.g., Baumol-Tobin or Sidrauski. This theory predicts a stable downward sloping relationship between real balances and interest rates. This relationship is apparent in the low-frequency data, e.g., those describing decade to decade movements. The second theory is the so called "liquidity effect," namely that a central bank's purchase of bonds, which increases the amount of money, creates a transitory but persistent decrease in interest rates. These patterns are apparent in high-frequency data, such as those used in the VAR literature for the identification of monetary shocks. This paper presents an analytically tractable model with segmented asset markets that unifies both ideas, displaying a stable long-run money demand, and explaining its short-term "instability" in terms of the liquidity effect. A new element of our model is a full characterization of the gradual propagation mechanism of monetary shocks in terms of a few structural parameters.

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We develop the model building on the ideas proposed by Grossman and Weiss 1983; Rotemberg (1984); Lucas (1990); Fuerst (1992); Christiano and Eichenbaum (1992); and others. In order to keep the analysis tractable these models, as well as the more recent search models using Lagos and Wright (2005) mechanism, abstract from the "lingering distributional effects" that follow an injection of liquidity.[1, 2] Our model is simple enough to include these effects and yet to allow us to completely characterize its solution, including a full analysis of the dynamic transmission of monetary shocks, in terms of three primitive parameters: a measure of the segmentation of the bond markets, and two parameters related to the households' intertemporal and intratemporal elasticity of substitution between real balances and consumption. A novel feature of the model is that as long as only a fraction of agents has access to open market operations, i.e., the bond market is segmented, the model displays a *persistent* liquidity effect even after a once-and-for-all increase in the money supply for any value of the parameters. The basic mechanism for the persistent liquidity effect is the redistribution of wealth that is caused by the operation. At the same time, and also novel in the segmented market models applied to monetary shocks, we show that if the changes in money growth are nearly permanent, the model delivers a standard money demand function with a positive relationship between velocity and interest rates. These three parameters dictate the long-run elasticity of money demand, the strength and persistence of the liquidity effect, and the amplitude of the interest rates response to monetary shocks.

We give an explicit characterization of the central bank open market operations in terms of money supply changes and a corresponding fiscal policy. While monetary and fiscal policy are always intertwined, we show that monetary policy has a distributional component in the presence of segmented asset markets, a role that typically belongs to fiscal policy. We find this characterization both useful for the construction of equilibrium, which gets simplified a lot, as well as for the substantive issue of interpreting open market operation as having a distributional impact toward the financial sector.

The model is useful to interpret several empirical facts about money demand. By combining the results for the dynamics of money growth, inflation and, interest rates, it provides a novel interpretation of the money demand data across low and high frequencies. Assuming that the money growth rate is driven by both permanent and transitory shocks, our theory implies that the low-frequency changes in monetary growth rates will generate a time series revealing an interest-elastic money demand such as the one estimated by Lucas (2000), Ireland (2009), and Lucas and Nicolini (2012). Instead, the high-frequency changes of money growth will generate data points that do not lie on this money demand curve because of the liquidity effect. This assumption allows us to reconcile low-frequency US data on interest rate and velocity, displaying

[1] See, for instance, Lucas (1990): "In [2] and [13], (referring to Grossman and Weiss (1983), and Rotemberg (1984)) an open market operation that induces a liquidity effect will also alter the distribution of wealth [...]. These distributional effects linger on indefinitely (as they no doubt do in reality), a fact that vastly complicates the analysis [...]. This paper studies this same liquidity effect using a simple device that abstracts from these distributional effects." Fuerst (1992) uses the same assumption and comments: "This methodology is not without cost. By entirely eliminating these wealth effects, the model loses the persistent and lingering effects of a monetary injection captured, for example, in Grossman and Weiss (1983)."

[2] The more recent "new monetary economics" literature likewise resorts to ingenious assumptions to achieve tractability. The quasi-linear preferences and the decentralized versus centralized markets a la Lagos-Wright make the equilibrium distribution of asset holdings degenerate. See Williamson and Wright (2010) for a review.

an interest-elastic money demand, with the high-frequency data, featuring persistent deviations from it and very small (even positive) interest elasticities.[3] Several authors have used high frequency data to document a persistent liquidity effect as well as a small interest elasticity after a monetary shock, see, for example, section 4.2.2 in Christiano, Eichenbaum, and Evans (1999) and our survey in online Appendix E. In Section IV, we interpret these facts using our model. In Section V, we evaluate the predictions of the model quantitatively by simulating a process for the growth rate of M1 compounding both a low- and a high-frequency component, and comparing the model predictions with the US data over the twentieth century.

Second, the predictions concerning the effects of transitory and persistent monetary shocks are useful to interpret the evidence in Sargent and Surico (2011) and Cogley, Sargent, and Surico (2011). They estimate that the correlation between interest rates and money growth is smaller than the one between inflation and money growth, and also show that these correlations vary from decade to decade.[4] They also provide an interpretation of these facts in terms of changes in the systematic response of monetary policy. Our model provides a complementary explanation of why, due to the liquidity effect, the correlation between money growth and interest rates is systematically smaller than the correlation between money growth and inflation.

Our model is closely related to the literature, both empirical and theoretical, where shocks to different "segments" of the asset market have mean reverting but persistent effects on relative prices because, due to frictions, capital moves slowly between them, e.g., Duffie (2010); Duffie and Strulovici (2012); Mitchell, Pedersen, and Pulvino (2007); or Edmond and Weill (2011). The model in this paper describes an economy with slow-moving capital and provides an analytical characterization of the link between the long-run behavior and the speed of adjustment. This mechanism can likewise be used to study the effects of shocks to the wealth of the financial intermediaries (ω, defined below), a topical issue in current research and policy discussions, and how those relative wealth shocks will transmit onto interest rates.

The Setup and Main Results.—Our model is a version of the segmented market model in Alvarez, Lucas, and Weber (2001) where, instead of using a binding cash in advance constraint, we use a Sidrauski money in the utility function set-up. Markets are segmented in the sense that only a fraction λ of households (the "traders") participates in the open market operations implemented by the central bank to control the money supply. Our model further allows for heterogeneity in the wealth of traders and nontraders as measured by ω, the ratio of the steady-state consumption of each trader relative to the average across all agents. We show that the effect of segmentation is completely summarized by the product of these variables: $\lambda\omega$, which measures the fraction of long-term wealth of the economy commanded by the traders.

[3] Indeed the reconciliation between the high- and low-frequency fluctuations of money demand is one of the challenges discussed by Lucas (2000, 250): "The interest elasticity needed to fit the long-term trends (and very sharply estimated by these trends) is much too high to permit a good fit on a year-to-year basis. Of course, it is precisely this difficulty that has motivated much of the money demand research of the last 30 years, and has led to distributed lag formulations of money demand that attempt to reconcile the evidence at different frequencies. In my opinion, this reconciliation has not yet been achieved [...]."

[4] See e.g., the first row of table 1 and the solid lines in figure 5 of Sargent and Surico (2011).

The household preferences are defined over a bundle $h(c, m)$ made of consumption c and real money balances m. The intratemporal substitution elasticity between c and m is ρ, the intertemporal substitution elasticity for h is γ. We show analytically how the three parameters $\lambda\omega$, ρ, γ determine the properties of the model steady-state and the economy's response, in terms of interest rates and inflation, to money supply shocks.

We begin by showing that velocity is a function of an appropriately expected discounted value of future monetary expansions, as occurs in the Sidrauski's model as well as in related monetary models (i.e., cash-credit, shopping time, etc). This implies that the price levels (and hence the inflation rate) are also functions of future expected paths of money (and hence of the growth of money). Surprisingly, the relationship between inflation and money is, up to a first order, exactly the same regardless of the degree of segmentation, as measured by λ and ω, and, hence, it is the same as in an otherwise standard monetary model. Instead, the equilibrium nominal interest rate is determined by the real trader's liquid asset, m/c, with an interest elasticity of $-\rho$.

The effects on interest rates of monetary shocks depend on whether they are temporary or permanent. We illustrate this result considering two extreme cases. The first is a permanent increase in the growth rate of money supply. This immediately increases expected inflation and produces a persistent increase in nominal interest rates and no liquidity effect. The second is a once-and-for-all increase of the money supply. In this case there is a jump on impact in the price level, and no effect on expected inflation. The model with segmented markets necessarily produces a persistent decrease in nominal interest rates: since the shock temporarily increases the m/c ratio of traders, they will only absorb the increase in money holdings at a lower interest rate. The size of the effect of a monetary expansion on interest rates is measured by the ratio of the total steady-state income of nontraders relative to the total steady-state income of traders, $(1 - \lambda\omega)/(\lambda\omega)$. More segmented markets produce bigger liquidity effects at all horizons because the monetary expansion, which has to be absorbed by traders, is larger relative to the wealth of the traders. The parameter λ and ω have no other effect on the response of interest rates. The persistence of the liquidity effect is increasing in ρ/γ, the ratio of the interest-elasticity of the money demand relative to the intertemporal substitution elasticity. The ratio ρ/γ governs how long it takes to the m/c ratio of traders to return to its steady-state value. Intuitively, the convergence of m/c is slow when the intertemporal substitutability of h is low (small γ), so that agents dislike fluctuations in the bundle of $h(c, m)$, or when m and c are good substitutes (large ρ), so that agents do not mind to have the m/c ratio different from its steady-state value.

For temporary but persistent shocks to the growth rate of the money supply the impulse responses of the nominal interest rate ranges between the two extreme cases described above, depending on the assumed persistence of the growth rate of money. We show analytically that impulse response of the nominal interest rate to a persistent shock is characterized by two eigenvalues. One is inherited from the persistence of the shock process itself. This component creates the classic "fisherian" response of the interest rate to changes in money growth and inflation. The other eigenvalue is related to the gradualism of the traders' adjustment rule for m/c, which was discussed above and is determined by ρ/γ. This component creates the "liquidity" effect which can, at least temporarily, dominate the fisherian effect.

Our model displays a liquidity effect for both unanticipated and anticipated monetary shocks, although these effects may differ. In the model in Lucas (1990), and in versions of Christiano and Eichenbaum (1992) and Fuerst (1992), only unanticipated shocks display liquidity effects. On the other hand, in Alvarez, Lucas, and Weber (2001); Occhino (2004); and Alvarez, Atkeson, and Kehoe (2002), there is no distinction between the effect of expected and unexpected monetary shocks on interest rates. The fact that both anticipated and unanticipated monetary shocks have an effect on interest rates, and that these effects differ, can be used in future applied work to estimate the impulse responses of interest rates by applying the identification strategy proposed by Cochrane (1998).

Related Literature on the Liquidity Effect.—The model in this paper is a descendent of the monetary models with segmented markets of Grossman and Weiss (1983) and Rotemberg (1984). These models are motivated by the hypothesis that, as described in Friedman's presidential address, and in contrast with the working of simpler neoclassical monetary models, an open market operation that increases the quantity of money once and for all produces a protracted decrease of the nominal interest rate.[5] In the models in these two papers agents are subject to a cash in advance constraint, but access to asset markets where open market operation takes place is restricted to a fraction of the agents. This fraction of agents, who holds only half of the money stock, have to absorb the entire increase on the money supply associated with the open market operation, which will be absorbed only with a lower real interest rate. Moreover, if this effect is large enough, the nominal interest rate decreases also. Mostly for tractability, these two papers assume that agents have access to the asset markets every other period. Not surprisingly, with this pattern of visits to the asset market, the effect of a once-and-for-all increase in money supply on interest rates are short lived. The largest effect is in the first two periods, after which there are small lingering echo effects.

Several monetary models of segmented asset markets have been written to analyze a variety of related questions since the seminal work of Grossman and Weiss, and Rotemberg. In all of them, some carefully chosen assumptions are used to avoid keeping track of the lingering effects on the cross-section distribution of asset holdings produced by an open market operation. The simplifications have the advantage of allowing a sharper analytical characterization of the equilibrium. Examples of these are Lucas (1990); Alvarez and Atkeson (1997); Fuerst (1992); Christiano and Eichenbaum (1992); Alvarez, Atkeson, and Kehoe (2002); Alvarez, Lucas, and Weber (2001); and Occhino (2004), which are discussed briefly in online Appendix E.2. All these models have in common that an open market operation that produces a once-and-for-all increase in the money supply decreases interest rate on impact, but

[5] For instance, pages 5 and 6 of Friedman's 1968 presidential address: "Let the Fed set out to keep interest rates down. How will it try to do so? By buying securities. This raises their prices and lowers their yields. In the process, it also increases the quantity of reserves available to banks, hence the amount of bank credit, and, ultimately the total quantity of money. That is why central bankers in particular, and the financial community more broadly, generally believe that an increase in the quantity of money tends to lower interest rates. [...] The initial impact of increasing the quantity of money at a faster rate than it has been increasing is to make interest rates lower for a time than they would otherwise have been. [...] after a somewhat longer interval, say, a year or two, [they will tend] to return interest rates to the level they would otherwise have had."

then the interest rate returns immediately to its previous level. These models also have in common that they produce a version of the quantity theory in which different permanent values of the growth rate of money supply are associated with different inflation rates, and, hence, via a Fisher equation, different nominal interest rates, but with the same level of velocity. In this sense, these models have an interest rate inelastic long-run money demand. In this paper, we introduce a modification of the setup in which a *once-and-for-all* increase in the money supply produces a persistent liquidity effect, due to the persistent redistribution effects of the open market operation, and additionally it implies a long-run, interest-elastic money demand.

There are a few models where a *once-and-for-all* increase in the money supply produces a persistent decline in interest rates. These models feature a different mechanism to generate persistence. An early example is the model in Christiano and Eichenbaum (1992). In their basic setup firms face a CIA constraint and, given the assumption on when the households have to decide their cash holdings, asset markets are segmented. In this setup, as well as in the closely related set-up of Fuerst (1992), liquidity effects are short lived.[6] Christiano and Eichenbaum (1992) add to this basic setup a convex adjustment cost applied to changes in the households' cash holdings. The adjustment cost naturally retards the adjustments, producing a persistent liquidity effect. Another example is Williamson (2008), who combines a segmented asset market model similar to Alvarez, Lucas, and Weber (2001), with persistent but mean reverting segmentation in the goods markets. In this setup, money spent by those connected to the asset market leaks slowly to the rest of the economy, spreading the effect of a *once-and-for-all* increase in money. A different mechanism proposed in the literature for persistent liquidity effects is sticky prices, á la Calvo. Several papers focus on the conditions under which liquidity effects emerge in simple versions of these models, such as Gali (2002) and Andrés, López-Salido, and Vallés (2002). A shortcoming of these models is that they are unable to generate a liquidity effect *and* to be consistent with the features of long-run money demand, such as a unit income elasticity and an interest elasticity around $-1/2$ (see online Appendix C for a more detailed discussion).

I. The Model

Let $U(c, m)$ be the period utility function, where c^i is consumption and m^i are beginning-of-period real balances. We assume that U is strictly increasing and concave. Let $i = T, N$ be the type of agents: traders, of which there are λ, and nontraders, of which there are $(1 - \lambda)$, respectively. While the focus of the paper is on the analysis of the behavior of interest rates due to the unequal access to asset market at the time of open market operations, we first describe the equilibrium in a

[6] Fuerst (1992) explains the lack of propagation in terms of his model very clearly: "As for the failures, the most glaring is that the model is lacking a strong propagation mechanism. The real effects of monetary injections are a result of (serially uncorrelated) forecast errors. These effects will therefore be strongest during the initial period of the shock."..." This failure to achieve persistence is more a criticism of my particular modeling strategy than of this class of models as a whole. If it takes more than one period for the economy to re-balance its portfolio and 'undo' the monetary injection, then the effects of monetary shocks will of course persist. While this assumption may be the most natural, it is also somewhat intractable..."

model without a bond market, and hence without open market operations. We do so because it is easier, and it highlights the logic of the determination of different equilibrium variables.

Time is discrete and starts at $t = 0$. The timing within a period is as follows: agents start with nominal beginning-of-period cash balances M_t^i, they receive nominal income $P_t\, y^i$, and choose real consumption c_t^i and end-of-period nominal balances N_t^i. Next period nominal cash balances, M_{t+1}^i, are given by this period nominal balances plus the nominal lump-sum transfer from the government, $P_{t+1}\, \tau_{t+1}^i$. Their budget constraint at $t \geq 0$ are

$$(1) \qquad N_t^i + P_t c_t^i = P_t y^i + M_t^i, \quad M_{t+1}^i = N_t^i + P_{t+1} \tau_{t+1}^i.$$

Our choice of the timing convention for this problem is standard and is consistent with an interpretation of U as a cash-credit good, as in Lucas and Stokey (1987). We use a Sidrauski money-in-the-utility function specification because, relative to a cash-in-advance model, it allows more flexibility to accommodate a stock m and a flow c, albeit in a mechanical way. We will return to the discussion of the specification, and the relation between stocks and flows, below.

The problem of an agent of type i is

$$\max_{\{N_t^i\}_{t=0}^{\infty}} E_0\left[\sum_{t=0}^{\infty} \beta^t U\left(c_t^i, M_t^i/P_t\right)\right],$$

subject to (1) given M_0. Notice that while we labeled the agents traders and nontraders, the budget constraint in (1) indicates that neither type of agent is allowed to trade in bonds or any other security. Their only intertemporal choice is the accumulation of cash balances. Yet, in Section IA, we show that the equilibrium of the model will be the same as one in which traders and the government participate in a market for nominal bonds.

Market clearing of goods and money is given by

$$\lambda c_t^T + (1 - \lambda)\, c_t^N = \lambda y^T + (1 - \lambda) y^N,$$

$$\lambda M_t^T + (1 - \lambda)\, M_t^N = M_t$$

for all $t \geq 0$. The government budget constraint is given by

$$M_t - M_{t-1} = P_t\left(\lambda \tau_t^T + (1 - \lambda)\, \tau_t^N\right)$$

for all $t \geq 1$. Notice that the government budget constraint does not apply for $t = 0$, since our timing convention is that we start the period with the cash after transfers. To simplify the notation, in this section we assume that the government does not trade in bonds, an assumption that we remove in Section IA. We note for future reference that the budget constraint of the agents and market clearing imply that aggregate beginning-of-period and end-of-period money balances are the same,

$$M_t = N_t \equiv \lambda N_t^T + (1 - \lambda) N_t^N,$$

for all $t \geq 0$. Notice that using the definition of N_t the budget constraint of the government follows from the budget constraint of the households:

$$\begin{aligned} M_{t+1} &= M_t + P_{t+1}\left[\lambda \tau_{t+1}^T + (1 - \lambda)\tau_{t+1}^N\right] \\ &= N_t + P_{t+1}\left[\lambda \tau_{t+1}^T + (1 - \lambda)\tau_{t+1}^N\right]. \end{aligned}$$

We define inflation π_t, growth rate of money, μ_t, beginning-of-period real balances, m_t, and end-of-period real balances, n_t as

$$\pi_{t+1} = \frac{P_{t+1}}{P_t}, \quad \mu_{t+1} = \frac{M_{t+1}}{M_t}, \quad m_t = \frac{M_t}{P_t}, \quad m_t^i = \frac{M_t^i}{P_t}, \quad n_t^i = \frac{N_t^i}{P_t},$$

for $i = T, N$, and $t \geq 0$. With these definitions we write the budget constraints as

$$(2) \qquad c_t^i + n_t^i = y^i + m_t^i \quad \text{and} \quad m_{t+1}^i = n_t^i/\pi_{t+1} + \tau_{t+1}^i, \quad t \geq 0,$$

market clearing as

$$(3) \qquad \begin{aligned} \lambda m_t^T + (1 - \lambda) m_t^N &= m_t, \\ \lambda c_t^T + (1 - \lambda) c_t^N &= \lambda y^T + (1 - \lambda) y^N, \\ t &\geq 0 \end{aligned}$$

the money-growth identity as

$$(4) \qquad \pi_t = \mu_t \frac{m_{t-1}}{m_t}, \quad t \geq 0,$$

and the government budget constraint as

$$(5) \qquad m_t - \frac{m_{t-1}}{\pi_t} = \lambda \tau_t^T + (1 - \lambda)\tau_t^N, \quad t \geq 1.$$

The exogenous random processes for this economy are given by $s_t \equiv (\mu_t, \tau_t^T, \tau_t^N)$. We use s^t for the histories of such shocks, but we avoid this notation when it is clear from the context. The first-order condition for the agent's problem with respect to n_t^i, $i = T, N$ is

$$(6) \qquad U_1(c_t^i, m_t^i) = E_t\left\{\frac{\beta}{\pi_{t+1}}\left[U_1(c_{t+1}^i, m_{t+1}^i) + U_2(c_{t+1}^i, m_{t+1}^i)\right]\right\},$$

for $t \geq 0$, where the expectation is with respect to the realization of inflation and the lump-sum subsidy τ^i_{t+1}. The perturbation corresponding to this Euler equation concerns a one-period increase in real balances. As in the case of a one period bond (with a real return equal to minus the inflation rate), the condition equates the current marginal utility of consumption with the discounted expected future utility of consumption plus, in this model with money in the utility function, the additional benefit of holding an extra amount of money for one period. We can now define an equilibrium:

DEFINITION 1: *Given initial conditions* $\{M^i_0\}$ *and a monetary and fiscal policy described by stochastic processes* $\{\mu_{t+1}, \tau^T_{t+1}, \tau^N_{t+1}\}^\infty_{t=0}$, *an Equilibrium is an initial price level* P_0, *inflation rates* $\{\pi_{t+1}\}^\infty_{t=0}$, *and stochastic processes* $\{n^i_t, m^i_t, c^i_t, m_t\}^\infty_{t=0}$ for $i = T, N$, *such that: the budget constraints* (2), *market clearing* (3), *identity* (4), *the government budget constraint* (5), *and the Euler equations* (6) *hold.*

We conclude this section with two comments on the setup. The first is that our convention for the timing of the model is one where the initial conditions M^i_0 contain the period zero monetary injection, and thus τ^i_0 and μ_0 are not part of the setup. For instance, our convention entails that $\{\mu_1, \tau^i_1\}$ are random variables, whose realization is not known as of time $t = 0$. Second, in the monetary-fiscal policy described by (5), the resources obtained by monetary expansions can be used for redistribution across agents, since the lump sum transfers τ^T_t and τ^N_t are allowed to differ across types.

A. *Interest Rates in an Equilibrium with an Active Bond Market*

In this section, we introduce a bond market in which traders participate to the open market operation. We show an equivalence result for the equilibrium with and without an active bond market, and analyze how the interest rate depends on the equilibrium allocation.

We assume that traders at time t have access to a bond market and a set of Arrow securities that pay contingent on the realization of $s_{t+1,}$ which opens at the beginning of the period, before utility is realized (including money holdings), at the same time of the money transfer τ^T_t. Thus, if a trader buys one nominal zero coupon bond in period t he reduces his money holdings M^T_t by Q_t dollars (the bond price), and increases next period holding of money in all states by \$1. Let W_t be the beginning-of-period money stock, after the current period lump-sum transfer from the government but before participating in the bond market. Let B^T_t be the number of nominal bonds purchased, and $A_t(s_{t+1})$ the quantity of Arrow securities that pay \$1 contingent on the realization of s_{t+1}. The price of each of these securities at the beginning of period t is denoted by $q_t(s_{t+1})$. In this case we can write the budget constraint of the trader at time t and history s^t, which we omit to simplify the notation, as

$$\sum_{s_{t+1}} A^T_t(s_{t+1}) q_t(s_{t+1}) \;+\; Q_t B^T_t \;+\; M^T_t \;=\; W_t,$$

$$N^T_t \;+\; P_t c^T_t \;=\; M^T_t \;+\; P_t y^T$$

$$W_{t+1}(s_{t+1}) \;=\; N^T_t \;+\; B^T_t \;+\; A^T_t(s_{t+1}) + \; P_{t+1}(s_{t+1})\,\tau^T_{t+1}(s_{t+1})$$

for $t \geq 0$. The interpretation is that during the period the agent chooses bond holdings B_t^T and consumption c_t^T, and given the budget constraint, this gives next-period cash balances before transfers N_t^T. The last line shows the beginning-of-next-period cash balances, which include the cash from the bonds purchased this period and the cash transfer from the government. Equivalently, we can write the budget constraint in real terms as

$$(7) \quad \sum_{s_{t+1}} a_t^T(s_{t+1}) q_t(s_{t+1}) + Q_t b_t^T + m_t^T = w_t,$$

$$n_t^T + c_t^T = m_t^T + y^T,$$

$$w_{t+1}(s_{t+1}) = \frac{n_t^T + b_t^T + a_t^T(s_{t+1})}{\pi_{t+1}(s_{t+1})} + \tau_{t+1}^T(s_{t+1}),$$

where $w_t = W_t/P_t$, $a_t = A_t/P_t$, $b_t = B_t/P_t$ and $t \geq 0$. The government budget constraint is

$$(8) \; \sum_{s_{t+1}} A_t(s_{t+1}) q_t(s_{t+1}) + Q_t B_t + M_t - M_{t-1} = B_{t-1} + A_{t-1} + P_t\left[\lambda \tau_t^T + (1-\lambda)\tau_t^N\right],$$

for $t \geq 1$ or in real terms:

$$(9) \; \sum_{s_{t+1}} a_t(s_{t+1}) q_t(s_{t+1}) + Q_t b_t + m_t = \frac{b_{t-1} + a_{t-1} + m_{t-1}}{\pi_t} + \lambda \tau_t^T + (1-\lambda)\tau_t^N,$$

for $t \geq 1$. The trader's problem is to maximize utility by choice of $\{n_t^T, b_t^T, a_t^T\}_{t=0}^{\infty}$ subject to (7), and given an initial condition w_0.

The first-order condition for the choice of n_t^T was given in (6). The first-order conditions for the choice of nominal bond holdings B_t^T and Arrow securities $A_t^T(s_{t+1})$ are

$$(10) \quad Q_t\left[U_1(c_t^T, m_t^T) + U_2(c_t^T, m_t^T)\right]$$

$$= E_t\left[\frac{\beta}{\pi_{t+1}}\left[U_1(c_{t+1}^T, m_{t+1}^T) + U_2(c_{t+1}^T, m_{t+1}^T)\right]\right]$$

$$q_t(s_{t+1})\left[U_1(c_t^T, m_t^T) + U_2(c_t^T, m_t^T)\right]$$

$$= \frac{\beta \Pr(s_{t+1}|s^t)}{\pi_{t+1}}\left[U_1(c_{t+1}^T, m_{t+1}^T) + U_2(c_{t+1}^T, m_{t+1}^T)\right],$$

where $\Pr(s_{t+1}|s^t)$ is the probability of state s_{t+1} conditional on the history s^t. As standard in the money-in-the-utility models, the difference between the first-order conditions for bonds, i.e., equation (10), and the one for money, i.e., equation (6), is that bonds have a price Q_t as well as a higher cost of acquiring them, since the agent can use money during period t, a feature captured by the term $U_2(c, m)$ in the left-hand side of equation (10).

Combining this first-order condition with the Euler equation (6), we obtain

$$Q_t = \frac{U_1(c_t^T, m_t^T)}{U_1(c_t^T, m_t^T) + U_2(c_t^T, m_t^T)}, \tag{11}$$

or, by letting r_t be the nominal interest rate, $1 + r_t \equiv Q_t^{-1}$, we can write

$$r_t = \frac{U_2(c_t^T, m_t^T)}{U_1(c_t^T, m_t^T)}.$$

Notice that, given the timing assumptions for the bond market, interest rates are functions of the time t allocation, i.e., they do not involve any expected future values. For instance consider the utility function

$$U(c,m) = \frac{h(c,m)^{1-1/\gamma} - 1}{1 - 1/\gamma}, \quad \text{where} \quad h(c,m) = \left[c^{1-1/\rho} + \mathcal{A}^{-1} m^{1-1/\rho}\right]^{\frac{\rho}{\rho-1}}, \tag{12}$$

which has a constant elasticity of substitution ρ between c and m, a constant intertemporal substitution elasticity γ between the consumption-money bundles, and is a parameter. This case yields the constant elasticity, unitary income, money demand

$$r_t = \frac{U_2(c_t^T, m_t^T)}{U_1(c_t^T, m_t^T)} = \frac{1}{\mathcal{A}} \left(\frac{m_t^T}{c_t^T}\right)^{-\frac{1}{\rho}}. \tag{13}$$

Finally, market clearing for Arrow securities and bonds, under the assumption that only traders participate in these markets, is

$$b_t = \lambda b_t^T \quad \text{and} \quad a_t(s_{t+1}) = \lambda a_t^T(s_{t+1}), \quad \forall s_{t+1} \tag{14}$$

for all $t \geq 0$. Next, we give an equilibrium definition for the model with an active nominal bond market:

DEFINITION 2: *Given initial conditions* $\{\tilde{M}_0^N, W_0\}$ *and a monetary and fiscal policy described by stochastic processes* $\{\tilde{\mu}_{t+1}, \tilde{\tau}_{t+1}^T, \tilde{\tau}_{t+1}^N, \tilde{b}_t, \tilde{a}_t\}_{t=0}^{\infty}$, *an Equilibrium with an active bond market is an initial price level* $\tilde{P}_0$, *inflation rate process* $\{\tilde{\pi}_{t+1}\}_{t=0}^{\infty}$, *stochastic processes* $\{\tilde{n}_t^i, \tilde{m}_t^i, \tilde{c}_t^i, \tilde{m}_t\}_{t=0}^{\infty}$ *for* $i = T, N$, *and stochastic processes* $\{w_t, b_t^T, a_t^T, q_t, Q_t\}_{t=0}^{\infty}$ *that satisfy: the budget constraints for nontraders* (2) *and traders* (7), *identity* (4), *Euler equations for end-of-period cash balances* (6), *the government budget constraint* (9), *the first-order condition for bonds and Arrow securities* (10), *and market clearing* (3) *and* (14).

As in the equilibrium described in Definition 1, our convention for the initial conditions W_0, M_0^N include the time zero money injection, and so neither τ_0^i nor μ_0 are part of the definition. Instead B_0^T and M_0^T are choices for the traders, and, hence, bond prices Q_t are determined starting from period $t = 0$ on. The following proposition shows the sense in which the equilibrium with and without an active bond market are equivalent (see online Appendix A for all proofs).

PROPOSITION 1: *Consider an equilibrium in the model without bond market:* $\langle P_0, \{\pi_{t+1}\}_{t=0}^{\infty}, \{n_t^i, c_t^i, m_t^i, m_t\}_{t=0}^{\infty}\rangle$, *for initial conditions* $\{M_0^i\}$ *and policy* $\{\mu_{t+1}, \tau_{t+1}^T, \tau_{t+1}^N\}_{t=0}^{\infty}$ *for* $i = T, N$. *Then, for any stochastic process of transfers to traders,* $\{\tilde{\tau}_{t+1}^T\}_{t=0}^{\infty}$, *there is an equilibrium with an active bond market that satisfies:*

$$\langle \tilde{P}_0, \{\tilde{\pi}_{t+1}\}_{t=0}^{\infty}, \{\tilde{n}_t^i, \tilde{c}_t^i, \tilde{m}_t^i, \tilde{m}_t\}_{t=0}^{\infty}\rangle = \langle P_0, \{\pi_{t+1}\}_{t=0}^{\infty}, \{n_t^i, c_t^i, m_t^i, m_t\}_{t=0}^{\infty}\rangle,$$

with bond and Arrow prices $\{Q_t, q_t\}_{t=0}^{\infty}$ *given by* (10) *for all* $t \geq 0$, *the fiscal and monetary policy given by* $\{\tilde{\tau}_t^N, \tilde{\mu}_t\}_{t=1}^{\infty} = \{\tau_t^N, \mu_t\}_{t=1}^{\infty}$, *and* $\{a_t, b_t\}_{t=1}^{\infty}$ *satisfying* (14) *and:*

$$\frac{a_{t-1} + b_{t-1}}{\tilde{\pi}_t} - Q_t b_t - \sum_{s_{t+1}} q_t(s_{t+1}) a_t(s_{t+1}) = \lambda(\tau_t^T - \tilde{\tau}_t^T), \quad t \geq 1,$$

with initial conditions $\tilde{M}_0^N = M_0^N$, $W_0 = M_0^T - Q_0(a_0 + b_0)/\lambda$, *where* $a_0 + b_0$ *satisfies an appropriately chosen present value.*

The proposition shows that it is only the combination of monetary and fiscal transfers that matters. While monetary and fiscal policy are always intertwined, as in the standard model with $\lambda = 1$, the fiscal policy that mimics an open market operation has a clear distributional component in the presence of segmented asset markets. We use this proposition to analyze an equilibrium where all monetary injections are carried out through open market operations. To see this, first consider an equilibrium without an active bond market and where $\tau_t^N = 0$. In this case, the budget constraint of the government is

$$M_t - M_{t-1} = \lambda P_t \tau_t^T.$$

Then, using the previous proposition, we can construct an equilibrium where $\tilde{\tau}_t^T = 0$ for all $t \geq 0$. The government budget constraint is

$$\sum_{s_{t+1}} q_t(s_{t+1}) A_t(s_{t+1}) + Q_t B_t + M_t - M_{t-1} = B_{t-1} + A_{t-1},$$

which, in the case of a time varying but deterministic policy, is

$$Q_t B_t + M_t - M_{t-1} = B_{t-1}, \quad t \geq 1 \qquad \text{or} \qquad \sum_{t=1}^{\infty} (M_t - M_{t-1})(\Pi_{i=1}^t Q_i) = B_0.$$

In this equilibrium traders start with an initial value of government bonds that enables them to buy the present value of the future seigniorage. The equivalence of real allocations, in the equilibrium where transfers differ across agents and in the one where money injections are carried out through open market operations, illustrates the sense in which open market operations with segmented asset markets have redistributive effects.

In Section IV we will consider a fiscal monetary policy that is similar to the one described above. We will assume $\tau_t^N = \tilde{\tau}_t^N = \bar{\tau}^N$ and $\tilde{\tau}_t^T = 0$. In this case nontraders receive a transfer with constant real value $\bar{\tau}^N$ and the government budget constraint is

$$\sum_{s_{t+1}} q_t(s_{t+1})A_t(s_{t+1}) + Q_t B_t + M_t - M_{t-1} = B_{t-1} + A_{t-1} + P_t(1 - \lambda)\bar{\tau}^N.$$

We will discuss the choice of the steady state $\bar{\tau}^N$ in Section II.

II. Approximate Aggregation with Segmented Markets

This section studies the determination of inflation in the model with segmentation, i.e., when the fraction of traders is $\lambda \in (0,1)$ and where each trader's income relative to the average in the economy is $\omega \in (0, 1/\lambda)$. Given the equivalence established by Proposition 1, the argument is developed using the simpler framework without an active bond market. In particular, we allow traders and nontraders to be subject to different arbitrary processes for $\{\tau_t^N\}_{t=1}^{\infty}$ and $\{\tau_t^T\}_{t=1}^{\infty}$. We show that, somewhat surprisingly, up to a linear approximation the relation between aggregate inverse velocity m_t/y and money growth rates $\{\mu_t\}_{t=1}^{\infty}$ is independent of the degree of segmentation measured by λ and ω, and, hence, that it is the same one obtained in a model with a representative agent, where $\lambda = 1$ and $\omega = 1$.

Consider the steady state with constant money growth at rate $\bar{\mu} = \bar{\pi}$ and a representative agent with constant real income y, so aggregate seignorage and real balances solve

$$\bar{\tau} = \bar{m}(\bar{\mu} - 1), \quad U_1(y, \bar{m}) = \frac{\beta}{\bar{\mu}}[U_1(y, \bar{m}) + U_2(y, \bar{m})].$$

We assume that per capita before tax real income for agents of each type is constant and given by

$$y^T = \omega y \qquad \text{and} \qquad y^N = \frac{1 - \lambda\omega}{1 - \lambda}\, y.$$

Furthermore we assume that the steady-state per capita share of seignorage of each type is proportional to their income share, i.e.

$$\text{(15)} \qquad \bar{\tau}^T = \omega\bar{m}(\bar{\pi} - 1) \qquad \text{and} \qquad \bar{\tau}^N = \frac{1 - \lambda\omega}{1 - \lambda}\bar{m}(\bar{\pi} - 1).$$

We define an approximate equilibrium by replacing the Euler equations of each agent and the budget constraints by linear approximations around the values that correspond to the steady-state consumption and real balances for each type. In this steady state, inflation satisfies $\overline{\mu} = \overline{\pi}$ and

$$\overline{m}^T = \omega\overline{m}, \quad \overline{c}^T = \omega\overline{c} \qquad \text{and} \qquad \overline{m}^N = \frac{1 - \lambda\omega}{1 - \lambda}\overline{m}, \quad \overline{c}^N = \frac{1 - \lambda\omega}{1 - \lambda}\overline{c}.$$

Let $\hat{x}_t \equiv x - \overline{x}$ denote the deviation of the variable x from its steady-state value $\overline{x}$. The linearization of the Euler equation (6) gives

$$(16)\quad \overline{U}^i_{11}\hat{c}^i_t + \overline{U}^i_{12}\hat{m}^i_t$$
$$= \frac{\beta}{\overline{\pi}}E_t\big[(\overline{U}^i_{11} + \overline{U}^i_{21})\hat{c}^i_{t+1} + (\overline{U}^i_{12} + \overline{U}^i_{22})\hat{m}^i_{t+1}\big] - \frac{\beta}{\overline{\pi}^2}E_t[\overline{U}^i_1 + \overline{U}^i_2]\hat{\pi}_{t+1},$$

where the derivatives U^i are evaluated at $(\overline{m}^i, \overline{c}^i)$. The linearization of the identity in (4), and the budget constraints in (2) and (5) give

$$(17)\qquad \hat{\pi}_t = \hat{\mu}_t - \frac{\overline{\mu}}{\overline{m}}(\hat{m}_t - \hat{m}_{t-1}),$$
$$\hat{m}^i_t = \frac{\hat{n}^i_{t-1}}{\overline{\pi}} - \frac{\overline{n}^i}{\overline{\pi}^2}\hat{\pi}_t + \hat{\tau}^i_t,$$
$$\hat{m}_t - \frac{\hat{m}_{t-1}}{\overline{\pi}} + \frac{\overline{m}}{\overline{\pi}^2}\hat{\pi}_t = \lambda\hat{\tau}^T_t + (1 - \lambda)\hat{\tau}^N_t,$$

for $i = N, T$. We are now ready to define an approximate equilibrium.

DEFINITION 3: *Given initial conditions* $\{M^i_0\}$ *and a fiscal and monetary policy described by* $\{\mu_{t+1}, \tau^i_{t+1}\}^\infty_{t=0}$ *an Approximate Equilibrium is given by* $\{n^i_t, m^i_t, c^i_t, m_t\}^\infty_{t=0}$ *for* $i = T, N$, *and* $P_0, \{\pi_{t+1}\}^\infty_{t=0}$ *that satisfy: market clearing* (3), *the linearized Euler equation* (16), *and the linearized constraints* (17).

Using this definition we state our main result on aggregation:

PROPOSITION 2: *In an approximate equilibrium the processes for aggregate real balances and inflation* $\{m_t, \pi_{t+1}\}^\infty_{t=0}$ *and the initial price level* P_0 *are the same for any* $\lambda \in [0, 1]$ *and any* $\omega \in (0, 1/\lambda)$.

The proof of this result relies on linearity (see online Appendix A for details). In the equilibrium (without active bond markets) traders' and nontraders' decisions are characterized by the *same* Euler equation, evaluated at different shocks for $\{\tau^i_t\}$, and the same inflation process. Linearizing these equations and using market clearing one obtains the aggregation result. This result is important for two reasons. First, substantively, it says that the relation between inflation and money growth, to a first-order approximation, is independent of the fraction of traders λ or their relative wealth ω_i. Second, it shows that the equilibrium has a recursive nature. One can determine first the path of aggregate real balances obtaining the process for inflation, as we do in Section III, and then solve for the decision problem of the nontrader, obtaining the process for the nontrader consumption and real balances. Using feasibility and the process for aggregate real balances, one can finally solve for the traders' real balances and consumption, which in turns gives us the interest rate from equation (11). Since the problem of the nontrader is a key intermediate step to determine the behavior of interest rates, Section IV analyzes it in detail. The reader who is not interested in the details of the derivation of the aggregate money demand, which due to the approximate aggregation are standard, can jump directly to Section IV. The relevant notation is summarized in equations (26)–(28).

III. Velocity and Money Growth

In this section, we consider a model with one type of agent, or $\lambda = 1$, to obtain a description of inverse velocity and inflation as functions of future expected money growth rates, as in the representative-agent model of Sidrauski, or in Cagan's. Our interest in the setup with $\lambda = 1$ comes from Proposition 2, which shows that the equilibrium path for aggregate inverse velocity and inflation is the same irrespective of $\lambda \in (0, 1)$.

Using market clearing ($c_t = y$) into the first order condition for m, and the inflation identity $\pi_{t+1} = \mu_{t+1}\, m_t/m_{t+1}$ we can write

(18) $$U_1(y, m_t)\, m_t = E_t\left\{\frac{\beta}{\mu_{t+1}}\left[U_1(y, m_{t+1}) + U_2(y, m_{t+1})\right] m_{t+1}\right\}.$$

Our next task is to analyze the behavior of this system. We first consider the steady state, the case where money supply grows at a constant rate $\bar{\mu}$ and $\bar{r}$ is the net interest rate that corresponds to a constant money growth rate and inflation $\bar{\mu}$:

(19) $$\frac{U_2(y, \bar{m})}{U_1(y, \bar{m})} = \bar{r} = \frac{\bar{\mu}}{\beta} - 1.$$

As in Lucas (2000) we interpret the function $\bar{m}$ of $\bar{r} = \bar{\mu}/\beta - 1$, solving equation (19), as the "long run" money demand. For the case where U is given by equation (12), this money demand has a constant interest rate elasticity $-\rho$.

In what follows, we analyze a *linearized* version of the difference equation (18), expanded around a constant $\mu = \bar{\mu}$ and $m = \bar{m}$. We seek a solution for real balances as a function of the future expected growth of the money supply.

PROPOSITION 3: *Let* $\hat{m}_t \equiv m_t - \bar{m}$, $\hat{\mu}_t \equiv \mu_t - \bar{\mu}$. *Linearizing* (18) *around* (19), *we have:*

(20) $$\hat{m}_t = \alpha E_t[\hat{\mu}_{t+1}] + \phi E_t[\hat{m}_{t+1}], \quad \text{where}$$

(21) $$\alpha \equiv -\frac{\bar{m}}{\bar{\mu}}\frac{\bar{U}_1}{\bar{U}_1 + \bar{m}\bar{U}_{12}} \quad \text{and} \quad \phi \equiv \frac{\beta}{\bar{\mu}}\left[1 + \frac{\bar{U}_2 + \bar{m}\bar{U}_{22}}{\bar{U}_1 + \bar{m}\bar{U}_{12}}\right],$$

and where $\overline{U}_i, \overline{U}_{ij}$ *are the derivatives of* $U(\cdot)$*evaluated at* $(y, \overline{m})$*. With* $0 < \phi < 1$*, we can express its unique bounded solution as*

$$\hat{m}_t = \alpha \sum_{i=1}^{\infty} \phi^{i-1} E_t[\hat{\mu}_{t+i}], \quad t \geq 0. \tag{22}$$

Thus, if $0 < \phi < 1$ and $\alpha < 0$, future expected money growth reduces current real money balances. We briefly discuss sufficient conditions for this configuration. Equation (21) shows that the condition for $\alpha < 0$ requires that $U_1 + mU_{12} > 0$, which is always the case if $U_{12} > 0$. The assumption of $U_{12} > 0$ has the interpretation that real balances are a complement to the consumption of nondurable consumption, and will be maintained for the rest of the paper. Notice that $-\alpha(\overline{\mu}/\overline{m})$ is decreasing in $U_{12} > 0$, starting from a value of 1 at $U_{12} = 0$.

When U is given by (12), the requirements for ϕ can be written in terms of conditions on: γ, ρ, $\bar{r}$, and $\overline{m}/y$. Lemma 1 in online Appendix A.A4 shows that when $\rho < \infty$, a sufficient condition for $\phi < 1$ and $\alpha < 0$ is

$$\frac{1}{\gamma} < 1 + \frac{1}{\rho} + \frac{y}{\bar{r} \cdot \overline{m}}, \tag{23}$$

(otherwise if $\rho = \infty$, then $\phi = 1$). Notice that condition (23) holds for a wide range of parameters of interest.[7] For instance, with an annual nominal interest rate of 4 percent, annual money-income ratio of 1/4, and an elasticity of substitution between consumption and real balances of 1/2, so that $\bar{r} = 0.04$, $\overline{m}/y = 1/4$, and $\rho = 1/2$, then γ has to be larger than 1/103, a condition that is satisfied by any reasonable estimate of the intertemporal elasticity of substitution γ.

Lemma 1 also shows that the condition $0 < \phi$, which ensures monotone dynamics, is:

$$\frac{1}{\gamma} < \frac{1}{1 + \bar{r}}\left(1 + \frac{1}{\rho} + \frac{y}{\bar{r}\,\overline{m}} + \bar{r} - \frac{(1/\rho - 1)y}{\overline{m}}\right),$$

which is implied by (23) as long as the length of a time period is sufficiently small.

A. *Linear State Space Representation for Velocity and Inflation*

We specify a linear time series process for $\{\hat{\mu}_t\}$ and rewrite the initial conditions exclusively in terms of real variables. We use a representation for inflation as a function of future money growth rates (as obtained from the linearization used in Proposition 3 and the initial aggregate money balances. Using (17) and (22), we obtain

$$\hat{\pi}_t = \hat{\mu}_t + \frac{\overline{\mu}}{\overline{m}} \hat{m}_{t-1} - \frac{\overline{\mu}}{\overline{m}} \alpha \sum_{i=1}^{\infty} \phi^{i-1} E_t[\hat{\mu}_{t+i}], \quad t \geq 0. \tag{24}$$

[7] Note that when U is given by (12) a sufficient condition for $U_{12} > 0$ is that the intertemporal elasticity of substitution, γ, is higher than the intratemporal elasticity of substitution between c and m, given by ρ.

Notice that equation (17) is defined for $\pi_0 \equiv P_0/P_{-1}$ and μ_0. This representation avoids us having to carry a nominal level variable, such as M_0, as the initial state. Instead, the initial state is the real level of money balances, $\hat{m}_{-1}$. We assume that the detrended growth of money supply $\hat{\mu}_t$ is a linear function of an exogenous state $\mathbf{z}_t$:

$$\hat{\mu}_{t+1} = \boldsymbol{\nu}\mathbf{z}_{t+1}, \quad \mathbf{z}_{t+1} = \boldsymbol{\Theta}z_t + \boldsymbol{\epsilon}_{t+1} \tag{25}$$

for $t \geq 0$, where $\mathbf{z}_0$ is given, $\boldsymbol{\nu}$ is a $k \times 1$ vector, $\boldsymbol{\Theta}$ is a $k \times k$ matrix with k stables eigenvalues, and $\boldsymbol{\epsilon}_{t+1}$ is a $k \times 1$ vector of innovations. In this case,

$$\hat{m}_t = \alpha \sum_{i=1}^{\infty} \phi^{i-1} E_t[\hat{\mu}_{t+i}] = \alpha \boldsymbol{\nu}\boldsymbol{\Theta}[I - \phi\boldsymbol{\Theta}]^{-1}\mathbf{z}_t, \quad t \geq 0.$$

Replacing (25) into (24) we obtain

$$\hat{\pi}_t = \boldsymbol{\nu}\mathbf{z}_t + \frac{\bar{\mu}}{\bar{m}}\hat{m}_{t-1} - \alpha\frac{\bar{\mu}}{\bar{m}}\boldsymbol{\nu}\boldsymbol{\Theta}[I - \phi\boldsymbol{\Theta}]^{-1}\mathbf{z}_t, \quad t \geq 0.$$

For example, if z_t is a scalar that follows the AR(1) process $z_{t+1} = \theta z_t + \epsilon_{t+1}$ and $\nu = 1$, then $k = 1$, $\hat{\mu}_t = z_t$, and $\Theta = \theta$, so that for $t \geq 0$:

$$\hat{m}_t = \alpha\frac{\theta}{1 - \phi\theta}\hat{\mu}_t, \qquad \hat{\pi}_t = \frac{\bar{\mu}}{\bar{m}}\hat{m}_{t-1} + \left[1 - \alpha\frac{\bar{\mu}}{\bar{m}}\frac{\theta}{1 - \phi\theta}\right]\hat{\mu}_t,$$

given $\hat{m}_{-1}$. Recall that $\alpha < 0$ if (23) holds, so that real balances are decreasing in $\hat{\mu}_t$ and, hence, inflation increases more than one for one with $\hat{\mu}_t$. This is a well-known feature of the standard variable-velocity model.

We summarize the linear equilibrium representation for the aggregate economy as a function of the innovations $\{\epsilon_t\}_{t=1}^{\infty}$, the parameters $\{\bar{\mu}/\bar{m}, \alpha, \phi, \boldsymbol{\nu}, \boldsymbol{\Theta}\}$, and initial conditions $\mathbf{z}_0$ and $\hat{m}_{-1}$ as follows:

$$\hat{m}_t = \boldsymbol{\kappa}\mathbf{z}_t, \quad \hat{\pi}_t = \left(\frac{\bar{\mu}}{\bar{m}}\right)\hat{m}_{t-1} + \boldsymbol{\zeta}\mathbf{z}_t, \qquad \mathbf{z}_{t+1} = \boldsymbol{\Theta}\mathbf{z}_t + \boldsymbol{\epsilon}_{t+1}, \tag{26}$$

for all $t \geq 0$, where the vectors $\boldsymbol{\zeta}$ and $\boldsymbol{\kappa}$ are given by

$$\boldsymbol{\zeta} \equiv \boldsymbol{\nu}\left(I - \alpha\frac{\bar{\mu}}{\bar{m}}\boldsymbol{\Theta}[I - \phi\boldsymbol{\Theta}]^{-1}\right), \quad \boldsymbol{\kappa} \equiv \alpha\boldsymbol{\nu}\boldsymbol{\Theta}[I - \phi\boldsymbol{\Theta}]^{-1}. \tag{27}$$

For future reference, we also provide a formula for expected inflation:

$$E_t[\hat{\pi}_{t+j}] = \overline{\boldsymbol{\Pi}}\boldsymbol{\Theta}^{j-1}\mathbf{z}_t, \quad \text{where} \quad \overline{\boldsymbol{\Pi}} \equiv \frac{\bar{\mu}}{\bar{m}}\boldsymbol{\kappa} + \boldsymbol{\zeta}\boldsymbol{\Theta}. \tag{28}$$

As a special case, let z_t be a scalar and assume that $\nu = 1$ and $\Theta = \theta = 0$, so that $\hat{\mu}_t$ is independently and identically distributed, we then have that real money balances are constant ($\kappa = 0$), inflation is equal to money growth ($\zeta = 1$), and, hence, expected inflation is constant ($\overline{\Pi} = 0$), or:

$$\kappa = \overline{\Pi} = 0, \qquad \text{and} \qquad \zeta = 1. \tag{29}$$

Finally, we examine the behavior of interest rates in the case of $\lambda = 1$ for the utility function in (12). Denoting $\hat{r}_t = r_t - \bar{r}$ and linearizing equation (13), we have: $\hat{r}_t = -(1/\rho)(\bar{r}/\overline{m})\,\hat{m}_t$. Replacing $\hat{m}_t$ by (26) and using $\overline{\mu} = \overline{\pi}$ to express shocks in percentage, we have:

$$\frac{\hat{r}_t}{\bar{r}} = -\frac{1}{\rho}\frac{\kappa\overline{\pi}}{\overline{m}}\frac{z_t}{\overline{\mu}}, \qquad t \geq 0. \tag{30}$$

For instance, in the scalar case, where $z_{t+1} = \theta z_t + \epsilon_{t+1}$ and $\nu = 1$, we have

$$\frac{\hat{r}_t}{\bar{r}} = \frac{1}{\rho}\left(\frac{-\alpha}{\overline{m}}\right)\frac{\theta\overline{\pi}}{1 - \phi\theta}\frac{\hat{\mu}_t}{\mu}. \tag{31}$$

Since $\alpha < 0$ under our maintained assumptions, interest rates move in the same direction than $\hat{\mu}_t$, and, hence, there is *no liquidity effect in the standard model*. Additionally, nominal interest rates inherit the persistence of $\hat{\mu}_t$: in the case where $\theta = 0$, so that $\hat{\mu}_t$ is independently and identically distributed, then the nominal interest rate is constant.

IV. Interest Rates with Segmented Markets

This section analyzes interest rates for the following fiscal-monetary policy. We consider a steady state, i.e., a value of $\overline{m}$ for the aggregate balances that corresponds to a constant money growth rate $\overline{\mu}$ (the unconditional mean of the process for money growth). We set the fiscal policy as follows: $\tau_t^N = \bar{\tau}^N$ as described in (15), and endow the traders with an initial bond position that allows them to buy the seignorage not allocated to the nontraders, as outlined in Section IA, while giving them no direct transfers: $\tau_t^T = 0$. In the absence of shocks, in a steady state, traders' and nontraders' allocations are proportional to their income. Yet, when there are shocks, traders must absorb all innovations to the money supply.

While our focus is on interest rates for simplicity, we analyze the equilibrium where there is no active bond markets, which in Section IA was shown to be equivalent to one where all the money injections are carried out through open market operations. In particular, as explained above, the nature of the equilibrium is recursive: we first solve for the process for inflation in the aggregate model, then we solve for the nontrader's problem, which is done in the next subsection, and finally, using the equivalence of the allocations with and without an active bond market, we solve for interest rates.

We characterize analytically the response of interest rates to money growth shocks with different persistence by computing a linear approximation of the equilibrium.

Proposition 8 gives a closed-form solution for the impulse response of interest rates to a monetary shock, and Proposition 9 specializes the formula for the case of an independently and identically distributed money growth rate shock, expressing all the coefficients in terms of structural parameters. Section IVB further analyzes the cases with persistent increases in money growth. These results allow us to discuss the conditions under which interest rate responses are Fisherian or display a liquidity effect, and if so, how persistent. Furthermore, we show that in the presence of segmentation the short-run interest elasticity of the money demand is much smaller than the long-run elasticity. Section IVB also defines the short- and long-term interest rate elasticity of money demand. Both the presence of a persistent liquidity effect after a monetary shock, and the small interest elasticity of money demand over the short run, have been documented by several authors, see, for example, section 4.2.2 in Christiano, Eichenbaum, and Evans (1999).

A. *The Nontrader Problem*

We consider the problem of a nontrader choosing $\{n_t^N\}_{t=0}^{\infty}$, facing a constant real lump-sum transfer $\bar{\tau}^N$, a given process for inflation $\{\pi_{t+1}\}_{t=0}^{\infty}$, and given initial condition $m_0^N = n_{-1}^N/\pi_0 + \bar{\tau}^N$. We start by studying the nontrader problem assuming inflation is constant at $\bar{\pi} \geq 1$. In this case, the state of the problem is given simply by n_{t-1}^N. We solve for the optimal decision rule $g(\cdot)$, that gives $n_t^N = g(n_{t-1}^N)$, and find conditions under which it has a unique steady state $\bar{n}^N = g(\bar{n}^N)$ that is globally stable. Furthermore, we characterize the local dynamics of this problem, i.e., the value of $g'(\bar{n}^N)$. In the second part of this section we use these results to characterize the solution of the linearized Euler equation when inflation follows an arbitrary process.

Assume inflation is constant at $\bar{\pi} > 1$, and consider the Bellman equation for the nontrader problem:

$$(32) \quad V(n) = \max_{0 \leq \tilde{n} \leq y^N + \bar{\tau}^N + n/\bar{\pi}} \left\{ U\left(y^N + \bar{\tau}^N + n/\tilde{\pi} - \tilde{n}, \quad n/\bar{\pi} + \bar{\tau}^N\right) + \beta V(\tilde{n}) \right\}.$$

The next proposition uses this equation to characterize the policy function $\tilde{n} = g(n)$, the uniqueness and stability of the steady state, and the value of $g'(\bar{n}^N)$, which is important to determine the speed of convergence to the steady state for the nontrader problem.

PROPOSITION 4: *Assume* $\bar{\pi} > 1, \bar{\tau}^N > 0$, *that U is strictly concave and bounded above, and* $0 < U_{12} < -U_{11}$. *Then the function* $g(\cdot)$ *is strictly increasing, it has a unique interior steady state* $\bar{n}^N = g(\bar{n}^N)$ *that is globally stable, with* $0 < g'(\bar{n}^N) < 1$.

Using the decision rule $g(\cdot)$ and the budget constraint, we can define the optimal consumption rule. For future reference, it turns out to be more convenient to use m (real cash balances after the transfer) as the state.

The budget constraint of the agent is given by $m^N = n^N/\pi + \bar{\tau}^N$ and $m^N + y^N = c^N + g(n^N)$. Thus, $c(m^N) \equiv m^N + y^N - g([m^N - \bar{\tau}^N]\pi)$ and $c'(m^N)$ is given by $1 - \pi\, g'(n^N)$. The elasticity of the ratio $m^N/c(m^N)$ with respect to m^N is:

$$(33)\quad \chi(m^N) \equiv \frac{m^N}{m^N/c(m^N)} \frac{\partial(m^N/c(m^N))}{\partial m^N} = 1 - \frac{m^N}{c(m^N)}(1 - \pi g'(n^N)).$$

We are interested in this elasticity because the interest rate response to money shocks depends on the changes of the m/c ratio. Equation (33) shows that the eigenvalue $g'(\bar{n}^N)$, determining the persistence of the response to monetary shocks, also determines the impact effect of the monetary shock χ. Since the nominal interest rate is proportional to the m/c ratio of traders, see equation (13), a zero value of χ yields no liquidity effect, a positive value yields a liquidity effect.

In the next proposition, we specialize the utility function to U CRRA and h CES, as described in equation (12), and characterize the slope $g'(\bar{n}^N)$ and the elasticity $\chi(\bar{m}^N)$.

PROPOSITION 5: *Assume $\bar{\pi} > 1$ and the utility function U with parameter $\mathcal{A}$ as given by equation (12), satisfying $0 < U_{12} < -U_{11}$. For any values of the triplet ρ, $r = \bar{\pi}/\beta - 1$ and $\bar{m}^N/\bar{c}^N = \bar{m}/\bar{c} > 1$, let $\mathcal{A}$ be such that $r = U_2/U_1$ evaluated at $\bar{m}/\bar{c}$. Then $g'(\bar{n}^N)$ and $\chi(\bar{m}^N)$ depend only on $\bar{m}/\bar{c}$, ρ/γ, $\bar{\pi}$, β. Moreover: $0 < g'(\bar{n}^N) < 1$, $0 \leq \chi(\bar{m}^N)$. Finally $g'(\bar{n}^N)$ and $\chi(\bar{m}^N)$ are increasing in the ratio ρ/γ. Moreover, $g'(\bar{n}^N)$ and $\chi(\bar{m}^N)$ are independent of ω and λ.*

This proposition is important because it characterizes the determinants of the persistence of a liquidity shock: high values of g' (say close to 1) will make the adjustment very slow and, hence, the effect of a shock very persistent. The proposition states that, somewhat surprisingly, the value of g' depends only on the ratio between the elasticities, ρ/γ, as opposed to the elasticities γ and ρ separately.[8] That g' and χ are independent of ω and λ, the relative long-run level of wealth of the nontraders, follows immediately from the homotheticity of preferences and our assumptions on y^N and $\bar{\tau}^N$. Further, the proposition establishes that g' is increasing in the ratio: ρ/γ, so that convergence is fast (i.e., g' small) when the intertemporal substitution elasticity is high, and/or the intratemporal elasticity is small. To see why this is the case, consider the behavior of an agent who starts with cash balances below the steady state $\bar{m}$. To reach the steady state, the agent must reduce consumption. If the reduction in consumption is large, then convergence to the steady state is fast, and the liquidity effect is short lived. It is easy to see that a fast adjustment could happen because of two reasons. First, if real balances and consumption are poor substitutes, so that $\rho \approx 0$ and $\rho/\gamma \approx 0$. In this case, the agent would like to keep m/c almost constant, which implies that g' is small. Alternatively, if the intertemporal substitution elasticity γ is very high and so $\rho/\gamma \approx 0$. Since the agent substitutes intertemporally very easily then the speed of convergence is high, or g' is small.

[8]Recall that γ is the intertemporal elasticity of substitution of the bundle h, and that ρ is the intratemporal elasticity of substitution between real balances and consumption.

We conclude this part with a comment on the role of the steady-state value of $\overline{m}/\overline{c}$ in Proposition 5: the dependence of the speed of convergence on the ratio m/c is not standard, but it should be clear in this context. If the stock of money is very small relative to consumption, the effect of starting with a stock below the steady state can be quickly corrected: if $\overline{m}/\overline{c} = 1$ then $g'(\overline{n}^N) = 0$, so the steady state is attained immediately. In other words, if the stock m is small relative to the flow c, it must be that the length of the model period is so big that it makes the analysis of convergence uninteresting. Indeed, in a continuous-time version of the nontrader problem, that deals more naturally with the stock/flow distinction, this condition is not needed.

So far we have analyzed the problem for a nontrader when inflation is constant. Now we move to the problem of the nontrader facing the (linearized) equilibrium process for inflation. We use the steady state $\overline{n}^N$ to define $\hat{n}_t^N$, the deviations of the end-of-period real cash balances $\hat{n}_t^N \equiv n_t^N - \overline{n}^N$. Notice that a bounded process $\{n_t^N\}_{t=0}^{\infty}$ satisfying the Euler equation (6) is a solution to the nontrader's problem. Now we are ready to state a characterization of the linearized solution to the nontraders' Euler equation. Replacing the budget constraint (2) into the Euler equation (6) for nontraders with constant lump sum transfers $\tau_t^N = \overline{\tau}^N$, and linearizing with respect to (c, m, π) around the values $(\overline{c}^N, \overline{m}^N, \overline{\pi})$ we obtain

$$(34) \qquad E_t[\hat{n}_{t+1}^N] = \xi_0 \hat{n}_t^N - \frac{1}{\beta}\hat{n}_{t-1}^N + \xi_1 \hat{\pi}_t + \xi_2 E_t[\hat{\pi}_{t+1}],$$

where the coefficients ξ_i are functions of the second derivatives of U evaluated at the steady state as well as β and $\overline{\pi}$, given by equation (A.10) in online Appendix A.7. The next proposition assumes that inflation is governed by the linearized equilibrium described in Section III.

ASSUMPTION 1: *The deviation of inflation and real balances from their steady-state value* $\{\hat{\pi}_t, \hat{m}_t\}_{t=0}^{\infty}$ *are given by the linear stochastic difference equation with exogenous driving shocks* $\{z_{t+1}\}_{t=0}^{\infty}$ *described by the matrix and vectors* $\{\boldsymbol{\Theta}, \boldsymbol{\zeta}, \boldsymbol{\kappa}\}$ *as detailed in equation* (26). *They imply that* $E_t[\hat{\pi}_{t+1}] = \overline{\boldsymbol{\Pi}}\, \mathbf{z}_t$ *as given in equation* (28).

The state for the dynamic program of the nontrader problem is given by the last period real balances $\hat{n}_{t-1}^N$, and the variables needed to forecast inflation, which given the linear representation in Section III, are $(\mathbf{z}_t, \hat{m}_{t-1})$. The next proposition characterizes the solution $\hat{n}_t^N = \hat{g}(\hat{n}_{t-1}^N, \mathbf{z}_t, \hat{m}_{t-1})$ for the linearized Euler equation.

PROPOSITION 6: *Assume that U is bounded from above, that* $0 < U_{12} < -U_{11}$, *and that the deviation of inflation and real balances from their steady-state value* $\{\hat{\pi}_t, \hat{m}_t\}_{t=0}^{\infty}$ *are given by Assumption 1. The unique bounded solution of the linearized Euler equation is given by*

$$(35) \qquad \hat{n}_t^N = \hat{g}(\hat{n}_{t-1}^N, \mathbf{z}_t, \hat{m}_{t-1}) = \varphi_0 \hat{n}_{t-1}^N + \boldsymbol{\varphi}_1 \mathbf{z}_t + \varphi_2 \hat{m}_{t-1},$$

where the φ_0 *coefficient satisfies:* $0 < \varphi_0 = g'(\overline{n}^N) < 1$, *and where the coefficients* φ_i, given in equation (A.11) in the online Appendix A.7 *are functions of the coefficients* ξ_i *of equation* (34), *the parameters* $\beta, \overline{\pi}, \overline{m}$, *and the coefficients* $\boldsymbol{\kappa}$, $\boldsymbol{\theta}$, $\boldsymbol{\zeta}$, and

$\overline{\pi}$ *in equation* (26)–(28). Finally, while φ_0 *is independent of* ω *and* λ, φ_1 *and* φ_2 *are proportional to* $(1 - \lambda\omega)/(1 - \lambda)$.

The result that the slope of the linear optimal policy $\hat{g}(\cdot)$ that solves the linearized Euler $\partial\hat{g}/\partial\hat{n} \equiv \varphi_0$ is the same as the slope at the steady state of the optimal decision rule for the nonlinear problem $g'(\overline{n}^N)$, is a standard one. As it is standard, φ_0 is the (stable) solution of a quadratic equation with coefficients defined by ξ_0 and β. That the relative wealth of a nontrader, $(1 - \lambda\omega)(1 - \lambda)$, scales φ_1 and φ_2 is a direct consequence of homotheticity of preferences; that φ_0 is independent of ω and λ follows because it is the *slope* of g. Recall that if the growth rate of money $\hat{\mu}_t$ is independently and identically distributed then real balances are constant (see equation (29)), inflation is independently and identically distributed, and, hence, expected inflation is constant or: $\Theta = \kappa = \overline{\Pi} = 0$, $\zeta = 1$. In this case, using our notation for $\hat{g}(\cdot)$ and the expression (A11) for the coefficients φ_i in online Appendix A.7, we obtain that

$$\varphi_1 = -\beta\varphi_0\xi_1 = -\varphi_0\frac{\overline{n}^N}{\overline{\pi}},$$

so that for all $t \geq 0$, we have $\hat{m}_t = 0$, and thus $\hat{n}_t^N = \varphi_0\left(\hat{n}_{t-1}^N - \frac{\overline{n}^N}{\overline{\pi}}\hat{u}_t\right) + \varphi_2\hat{m}_{t-1}$. If the economy starts with $\hat{m}_{-1} = 0$:

$$\hat{n}_t^N = g'(\overline{n}^N)\,\hat{n}_{t-1}^N - g'(\overline{n}^N)\,\frac{\overline{n}^N}{\overline{\pi}}\,\hat{\mu}_t,$$

so $\hat{n}_t^N$ follows an autoregressive process of order one, with parameter $g'(\overline{n}^N)$, and innovations that are proportional to inflation $\hat{\pi}_t = \hat{\mu}_t$ with a (negative) coefficient given by $-g'(\overline{n}^N)\overline{n}^N/\overline{\pi}$.

As a summary, we describe the evolution of the state of the economy and its dynamics in the next proposition.

PROPOSITION 7: *The state of the* (*linearized*) *equilibrium for economy at time t is* $(\hat{n}_{t-1}^N, \hat{m}_{t-1}, \mathbf{z}_t)$. *The law of motion is given by equations* (26) *for* $(\hat{m}_t, \mathbf{z}_t)$ *and* (35) *for* $\hat{n}_t^N$. *The eigenvalues of the system are given by those on the exogenous state* $\mathbf{z}_t$, *i.e., the eigenvalues of the matrix* $\boldsymbol{\Theta}$, *and by the pair* $(0, \varphi_0)$.

The zero endogenous eigenvalue comes from the long-run money demand, since we can write $\hat{m}_t = 0\,\hat{m}_{t-1} + \boldsymbol{\kappa}\mathbf{z}_t$. The other endogenous stable eigenvalue $\varphi_0 = g'$ comes from the dynamics of the nontrader's problem and depends only on the ratio of the elasticities. Thus, the dynamics of any equilibrium variable, including interest rates, depends only on these eigenvalues.

B. *Interest Rates in a Linearized Equilibrium*

In this section, we use the aggregation result of Section II, the inflation dynamics of Section III and the characterization of the nontrader's dynamic problem of Section IVA to solve for the effect of open market operations on interest rates.

We are interested in the following particular monetary-fiscal policy. Nontraders receive a constant real transfer per period, equal to the steady-state value of seigniorage. Traders receive the remaining part of the seigniorage. The deviation from the steady-state growth of money supply evolves according to $\hat{\mu}_t = \boldsymbol{\nu} \mathbf{z}_t$, for an exogenous process $\mathbf{z}_t$, as described in (25). Equivalently, as shown in Section IA, we can regard this equilibrium as one in which nontraders receive a constant real tax rebate τ_t^N and traders receive no lump-sum rebate, but participate in open market operations. Returning to the equilibrium without an active bond market, the values of τ_t^N and τ_t^T are given as follows. Let $\bar{m}(\bar{\mu} - 1)/\bar{\mu}$ be the average seigniorage

$$\tau_t^N = \bar{\tau}^N \equiv \frac{\bar{m}(\bar{\mu} - 1)}{\bar{\mu}} \frac{1 - \lambda\omega}{1 - \lambda} \quad \text{and} \quad \tau_t^T = \frac{1}{\lambda}\left[\frac{M_t - M_{t-1}}{P_t}\right] - \frac{(1 - \lambda)}{\lambda} \bar{\tau}^N,$$

where $\bar{m}$ solves $\frac{U_2(y, \bar{m})}{U_1(y, \bar{m})} = \bar{r} \equiv \frac{\bar{\mu}}{\beta} - 1$. In steady state (i.e., when $\mu_t = \bar{\mu}$ all t), the value of τ_t^T is also constant, and, hence, $\tau_t^T = \omega \bar{m}(\bar{\mu} - 1)/\bar{\mu}$. It is straightforward to verify that these choices satisfy the government budget constraint (5). Also it is easy to verify that with these choices for τ_t^N and τ_t^T if M_t grows at a constant rate $\bar{\mu}$, then traders and nontraders will have the same consumption and money holdings as in (15).

Now we turn to the determination of the path of interest rates. To do so, let's use a first-order approximation around $\mu_t = \pi_t = \bar{\mu}$ and $m_t = \bar{m}$, $m_t^i = \bar{m}^i$, $c_t^i = \bar{c}^i$ for $i = N, T$, where $\hat{r}_t = r_t - \bar{r}$. Linearizing the first-order condition of the traders (11) with respect to c^T and m^T, replacing $\hat{c}_t^T$, $\hat{m}_t^T$ using market clearing for goods and money equation (3) to write the resulting expression in terms of $\hat{c}_t^N$, $\hat{m}_t^N$, and using that the elasticity of substitution between m and c is given by ρ, we obtain

$$\frac{\hat{r}_t}{\bar{r}} = -\frac{1}{\rho}\left(\frac{\hat{m}_t^T}{\bar{m}^T} - \frac{\hat{c}_t^T}{\bar{c}_t^T}\right) = -\frac{1}{\rho\lambda\omega}\left(\frac{\hat{m}_t}{\bar{m}} - (1 - \lambda\omega)\left(\frac{m_t^N}{\bar{m}^N} - \frac{\hat{c}_t^N}{\bar{c}^N}\right)\right).$$

Finally, we use this equation to solve for interest rates as follows. The term $\hat{m}_t$ is determined by the equilibrium in the aggregate economy, i.e., $\hat{m}_t = \boldsymbol{\kappa} \mathbf{z}_t$. The terms $\hat{m}_t^N$, $\hat{c}_t^N$ are determined by the solution of the nontrader problem. Using the budget constraint of the nontrader, we have $m_t^N = n_{t-1}^N/\pi_t + \bar{\tau}^N$ and $c_t^N = \bar{c}^N + n_{t-1}^N/\pi_t + \bar{\tau}^N - n_t^N$. Linearizing these expressions gives

$$\frac{\hat{m}_t^N}{\bar{m}^N} - \frac{\hat{c}_t^N}{\bar{c}^N} = \hat{n}_{t-1}^N \frac{1}{\bar{\pi}}\left(\frac{1}{\bar{m}^N} - \frac{1}{\bar{c}^N}\right) + \frac{\bar{n}^N}{\bar{\pi}^2}\left(\frac{1}{\bar{c}^N} - \frac{1}{\bar{m}^N}\right)\hat{\pi}_t + \frac{1}{\bar{c}^N}\hat{n}_t^N.$$

Using the decision rule of nontraders: $\hat n^N_t/\bar n^N = \varphi_0 \hat n^N_{t-1}/\bar n^N + \boldsymbol{\varphi}_1/\bar n^N \mathbf{z}_t + \varphi_2/\bar n^N \hat m_{t-1}$, to replace $\hat n^N_t$, and that inflation dynamics are given by $\hat\pi_t = \left(\frac{\bar\mu}{\bar m}\right)\hat m_{t-1} + \boldsymbol{\zeta}\mathbf{z}_t$ and $\hat m_t = \boldsymbol{\kappa}\mathbf{z}_t$, we obtain the following result:

PROPOSITION 8: *Interest rates evolve according to:*

$$(36)\quad \frac{\hat r_t}{\bar r} = -\frac{1}{\rho}\frac{\kappa\bar\pi}{\bar m}\frac{\mathbf{z}_t}{\bar\mu}$$

$$- \frac{1}{\rho}\frac{(1-\lambda\omega)}{\lambda\omega}\left[\frac{\bar n}{\bar m\bar\pi}\left(\left(1-\frac{\bar m}{\bar c}\right)\zeta - \frac{\bar m}{\bar c}\frac{\bar\pi^2\varphi_1}{\bar n^N}\right) + \frac{\kappa\bar\pi}{\bar m}\right]\frac{\mathbf{z}_t}{\bar\mu}$$

$$+ \frac{1}{\rho}\frac{(1-\lambda\omega)}{\lambda\omega}\frac{\bar n}{\bar m\bar\pi}\left[\left(1-\frac{\bar m}{\bar c}+\frac{\bar m}{\bar c}\bar\pi\varphi_0\right)\frac{\hat n^N_{t-1}}{\bar n^N} + \left(\frac{1}{\bar c} - 1 + \frac{\bar m\bar\pi\varphi_2}{\bar n\,\bar c^N}\right)\hat m_{t-1}\right],$$

where $\{\mathbf{z}_t\}$ *evolves according to equation* (25), $\{\hat m_t\}$ *evolves according to equation* (26), *and* $\{\hat n^N_t/\bar n^N\}$ *evolves according to equation* (35). *The only dependence of the evolution of interest rates on the segmentation parameters* ω *and* λ *is given by the term* $(1-\lambda\omega)/\lambda\omega$.

The first term in equation (36) is the interest rate that obtains in the absence of segmentation ($\lambda\omega = 1$), which was discussed in equation (30) and in which, as we noticed, there is no liquidity effect. The "liquidity effect," the difference compared to the nonsegmented case, is given by the terms in the second and third line of the expression. The interest rate is a function of $\left(\hat n^N_{t-1}/\bar n^N, \mathbf{z}_t, \hat m_{t-1}\right)$. The variables $\hat m_{t-1}$ and $\hat n^N_{t-1}/\bar n^N$ contain all the information needed to compute the time path of the distributional effects between traders and nontraders. Note that, as shown in Proposition 6, the ratios $\varphi_2/\bar c^N$ and $\boldsymbol{\varphi}_1/\bar n^N$ do *not* depend on λ or ω. For the same reasons, the law of motion of $\hat n^N_t/\bar n^N$ is also independent of ω and λ. As explained in Section III, the law of motion for the aggregate variables $\hat m_t$ is independent of λ and ω. Thus, the only dependence of these two parameters indexing segmentation on the evolution of interest rates is given by the ratio $(1-\lambda\omega)/\lambda\omega$, which measures the steady state ratio of the total wealth of nonmarket participants to the wealth of market participants. This result simplifies substantially the comparative statics with respect to the degree of segmentation: different steady-state wealth ratios scale the "liquidity effect."

Unexpected Once-and-for-All Increase in the Money Supply.—This section studies the impulse-response of nominal interest rates when the growth rate of money supply follows an independently and identically distributed process. Equivalently, we analyze the effect of starting the system at the steady state corresponding to $\bar\mu$ and then shock it with an unexpected transitory one time increase in the growth rate of the money supply at t, i.e., $\mu_t > \bar\mu$ and $\mu_{t+s} = \bar\mu$ for all $s \geq 1$, i.e., a once-and-for-all permanent increase in the level of the money supply.

Let the initial conditions $\hat{m}_{t-1} = \hat{n}^N_{t-1} = 0$. If $\hat{\mu}_t$ is independently and identically distributed we have that, as shown in (29), $\kappa = \Theta = 0$, $\nu = \zeta = 1$. This gives

$$\hat{\pi}_t = \hat{\mu}_t > 0, \quad \hat{\pi}_{t+s} = \hat{\mu}_{t+s} = 0 \quad \text{all} \quad s \geq 1, \quad \text{and} \quad \hat{m}_{t+s} = 0, s \geq 0.$$

For the nontraders, we have that for all $s \geq 0$:

$$\hat{n}^N_{t+s} = \varphi_0 \hat{n}^N_{t+s-1} + \varphi_1 \hat{z}_{t+s} + \varphi_2 \hat{m}_{t+s-1} = \varphi_0 \hat{n}^N_{t+s-1} = \varphi_0^s \hat{n}^N_t = \varphi_0^s \varphi_1 z_t.$$

Using $\bar{\pi} = \bar{\mu}$ and for independently and identically distributed shocks $\varphi_1 = -\varphi_0 \frac{\bar{n}^N}{\bar{\pi}}$ we derive the following proposition, which assumes that $m/c > 1$ (a condition related to the choice of time units discussed in the comment to Proposition 5).

PROPOSITION 9: *The effect of an unexpected once-and-for-all increase in the money supply at time t of size* $(\mu_t - \bar{\mu})/\bar{\mu}$ *is to decrease interest rates on impact, and gradually return to the steady state value* $\bar{r}$*, according to*

$$\frac{\hat{r}_{t+s}}{\bar{r}} = -\frac{1}{\rho} \frac{(1 - \lambda\omega)}{\lambda\omega} \frac{\bar{n}}{\bar{m}\bar{\pi}} \chi(\bar{m}) \varphi_0^s \frac{\hat{\mu} t}{\mu} \tag{37}$$

for all $s = 0, 1, 2, \ldots$, *where* $\chi(\bar{m}) = 1 - \frac{\bar{m}}{y}(1 - \bar{\pi}\varphi_0)$.

Equation (37) shows that the sign and persistence of the liquidity effect depend on the magnitude of φ_0. The impact effect is negative, i.e., the nominal interest rate decreases when money increases, if $\chi(\bar{m}) > 0$ a condition established in Proposition 5.

To understand the mechanics of the liquidity effect, note that the effect of an independently and identically distributed shock to money supply to the nontrader is to increase the price level, thus decreasing the post-transfer real money balances m of the nontrader. If the consumption elasticity is smaller than one, then the ratio of money to consumption for the nontraders decreases, i.e., $\chi(\bar{m}) > 0$. Since with an independently and identically distributed shock aggregate real balances remain the same, this implies that the ratio of money to consumption must increase for traders. In turn, a higher m/c ratio for traders implies, by equation (11), that the nominal interest rate must decrease.

The decrease on impact of the interest rate after a once-and-for-all increase in money increases with $\chi(\bar{m}) > 0$. Recall that Proposition 5 establishes that χ is an increasing function of ρ/γ. The persistence of the liquidity effect also depends on the magnitude of $\varphi_0 = g'(\bar{n}^N)$, which is also increasing in ρ/γ. The closer the value of φ_0 is to one, the more persistent the liquidity effect is. Finally, more segmented markets have larger amplitude of variation on interest rates. The strength of the segmentation is measured by the ratio of steady-state wealth of the nontraders relative to steady-state wealth of the traders, i.e., $(1 - \lambda\omega)/(\lambda\omega)$. When this ratio becomes larger, the liquidity effect is larger at all horizons. The parameters λ and ω do not enter in any of the other terms in this impulse response.

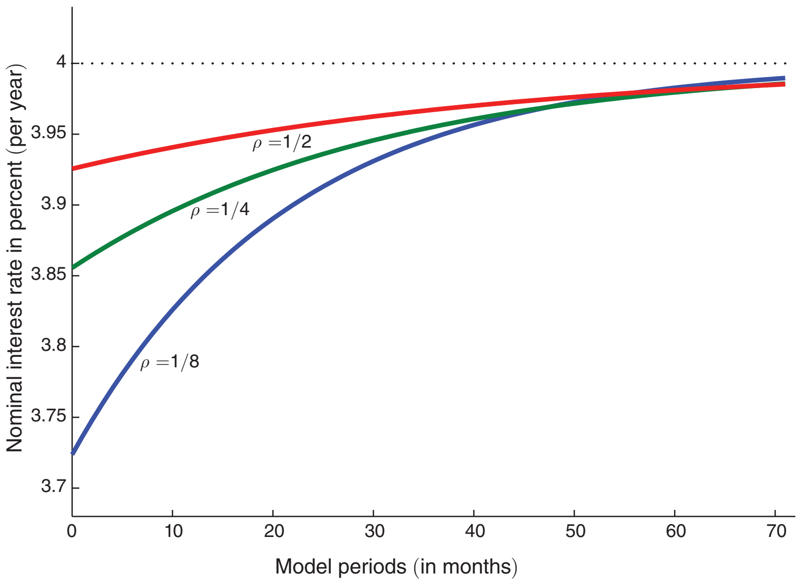

Figure 1. Response to a Once-and-for-All Money Supply Shock

Notes: The shock is an unanticipated increase of money causing a 1 percent increase of the price level. The other parameters are $\gamma = 1/4$, $\lambda\omega = 0.5$.

To illustrate this proposition, we compute the impulse response to a once-and-for-all shock to the money supply for different parameter values. In all the impulse responses we use annual inflation and real rate of 2 percent, U given by (12) and choose the value of the parameter to obtain a steady-state value of $\overline{m}/\overline{c}$ equal to 0.25 at annual frequency. We let the model period to be a month. In Figure 1, we plot the impulse response for a once-and-for-all shock to the money supply which implies a 1 percent increase in the price level on impact.[9] In the figure, we use $\lambda\omega = 0.5$, $\gamma = 1/4$ and we vary the value of ρ. As it is clear from Proposition 9, different values of $\lambda\omega$ scale the distance to the steady state by the same proportion at all horizons.

Figure 1 shows that for lower elasticity of the long-run money demand ρ, there are smaller liquidity effects at impact, with shorter lifetimes. The value of ρ has two opposite effects on the impulse response of interest rates, as can be seen from (37). The first is a direct effect of the preferences. A smaller value of the elasticity of substitution ρ means that for a given change in the money-consumption ratio m/c of traders there is a larger effect effect on interest rates. The second effect operates through the equilibrium determination of the elasticity χ, that was discussed in Proposition 5. For a fixed value of γ smaller values of ρ decrease χ and, hence, imply a smaller decrease at impact on the ratio m/c. The impulse responses in Figure 1 show that the first effect almost completely dominates the second one, since the vertical distance between the impulse responses at $t = 0$, is almost proportional to

[9] In the case considered in this section, where the money supply follows an independently and identically distributed process, this requires shocking the money supply by the same amount, namely $(\mu_0 - \overline{\mu})/\overline{\mu} = 0.01$, see Section IIIA for details.

Table 1—Once-and-for-All Shock to μ: Half-Life (τ, in Years), and Elasticity of m/c (χ)

γ ρ	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$
$\frac{1}{2}$	$\chi = 0.91$ $\tau = 1.77$	$\chi = 0.87$ $\tau = 1.25$	$\chi = 0.82$ $\tau = 0.89$
$\frac{1}{4}$	$\chi = 0.94$ $\tau = 2.53$	$\chi = 0.91$ $\tau = 1.77$	$\chi = 0.87$ $\tau = 1.25$
$\frac{1}{8}$	$\chi = 0.96$ $\tau = 3.63$	$\chi = 0.94$ $\tau = 2.53$	$\chi = 0.91$ $\tau = 1.77$

Note: $\tau = \log(1/2)/\log(\varphi_0)$ half-life of the interest shock, $\chi(\overline{m}) = 1 - \frac{\overline{m}}{c}\frac{dc}{dm}$ elasticity of $\frac{m}{c}$ with respect to m on impact.

the change in the value of $1/\rho$. Additionally, different values of ρ correspond to different persistence of the liquidity effect, through changes in $g'(\bar{n}^N)$. Lower values of ρ, as shown in Proposition 5, imply faster convergence, as the figure shows.

Table 1 complements Figure 1 by computing two of the determinants of the impulse response of interest rates after a once-and-for-all change in the money supply for different combinations of intertemporal elasticity of substitution γ, and intratemporal elasticity of substitution ρ. The half-life of the shock is a simple transformation of φ_0 given by $\tau \equiv [\log(1/2)/\log(\varphi_0)]$, expressed in years. The other determinant, denoted by χ, is the impact elasticity of m/c with respect to a change in m, which is also a simple function of φ_0.

The values for ρ and γ for Table 1 are chosen so that the ratio ρ/γ is constant on the diagonal. The range of values of ρ and γ are chosen to bracket most empirical estimates of the interest rate elasticity and of the intertemporal elasticity of substitution. The values of τ and χ across the diagonal of Table 1 are the same, which follows from Proposition 5, where it is shown that the decision rules $c(\cdot)$ and $g(\cdot)$ are functions of the ratio of the elasticities ρ/γ. The value of χ has the interpretation of the elasticity of the ratio m/c with respect to an unanticipated once-and-for-all shock to the price level (hence to m). The values for this elasticity varies between 0.82 and 0.95 across the values of ρ/γ reported in the table. The range of half-lives across the values of ρ/γ reported in Table 1 is between a bit less than a year, to more than 3.5 years.

In online Appendix D, we compare the approximate linear solution with the numerical solution of the nonlinear system. It is shown that for the examples considered in this section the approximation is very precise.

Persistent Increase in the Growth Rate of the Money Supply.—This section analyzes the effect of a persistent increase in the growth rate of the money supply on interest rates. We use the general expression for r_t in (36), the evolution of the state m_t given by (26), and the evolution of n_t given by (35).

Figure 2 plots the impulse responses of interest rates to money shocks under the assumption that the growth rate of the money supply is an AR(1) with autocorrelation θ, as opposed to independently and identically distributed. Otherwise the parameters are the ones used in Figure 1 in the case where $\rho = 1/2$ (so the long run elasticity of

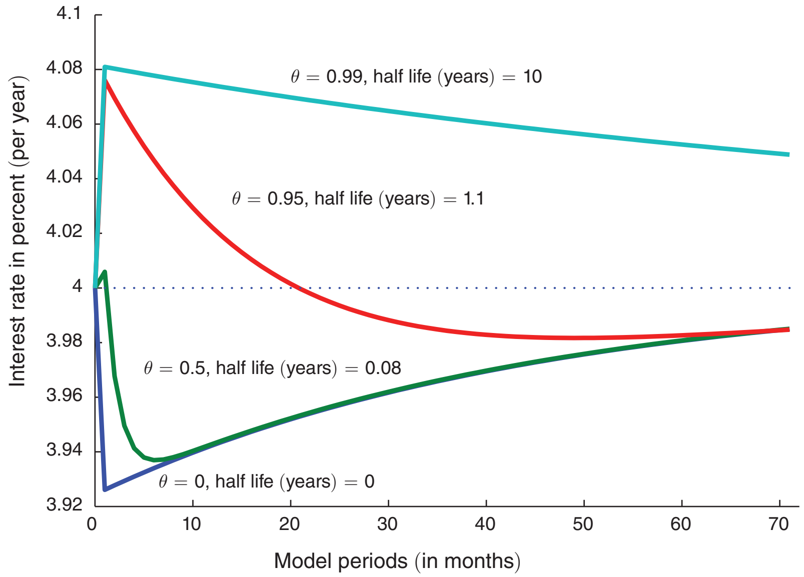

Figure 2. Response to a Persistent Increase in Money Supply

Notes: The shock is an unanticipated increase of money, causing a 1 percent increase of the price level. The other parameters are $\gamma = 1/4$ and $\lambda\omega = 0.5$.

the money demand is $1/2$). We plot the impulse response for four values of θ, corresponding to a half life of 0 months, 1 month, 13 months, and 10 years. The zero half life coincides with the independently and identically distributed case of Figure 1, and is included to help in the comparisons. The size of the initial shock to money is chosen so that the effect on the price level on impact is an increase of 1 percent, as in the case of independently and identically distributed money growth.

As can be seen from the impulse responses in Figure 2, the monetary shocks with a shorter half-life produce a liquidity effect. If the monetary shock is very persistent, instead, the Fisherian aspects of the model take over, expected inflation rises considerably on impact, and there is no liquidity effect. Notice that for intermediate values of θ the impulse response has a hump shape, attaining an extreme some periods after the impact effect. The hump shape of the impulse response is due to the fact that the dynamic system has two eigenvalues: θ, governing aggregate real balances and inflation; and φ_0, governing the nontraders adjustment of their real balances. Note that for the three smallest values of θ, the impulse responses eventually converge to the same line, since the short-run behavior is dominated by θ and the long-run behavior by φ_0. Instead, for the case where θ is near one, the Fisherian effect dominates and the impulse response is almost identical to the one where markets are not segmented.

To identify the effect of segmented markets on interest rates, Figure 3 displays the impulse response (to the same shocks) for the model with $\lambda = 1$, which has no liquidity effects (Section III). When shocks are short lived, so that there are no movements in the expected growth rate of money, interest rates remain almost constant at the steady-state level. Comparing Figure 2 and 3 shows that when monetary shocks are

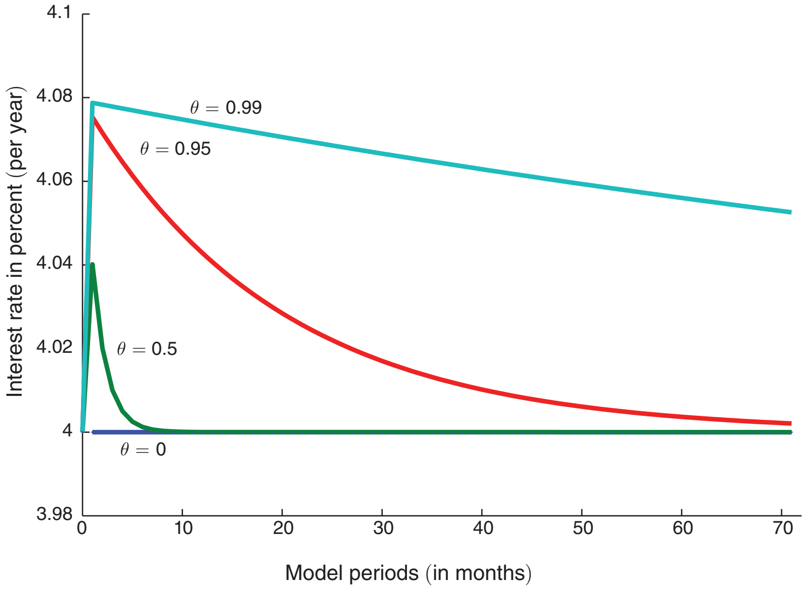

Figure 3. Response with No Segmentation ($\lambda = 1$)

Notes: The shock is an unanticipated increase of money causing a 1 percent increase of the price level. The other parameters are $\gamma = 1/4$, $\lambda = 1.0$ (i.e., no segmentation).

very persistent the behavior of interest rates in the model with segmented markets ($\lambda\omega = 0.5$) is similar to the one in the model with homogenous agents ($\lambda = 1$).

In online Appendix D, it is shown that the approximate linear solution is very close to the exact numerical solution of the nonlinear system for all experiments displayed in Figure 2 and Figure 3.

Short- versus Long-Run Money Demand Elasticities and the Liquidity Effect.— We conclude the section with a comment on the relation between the liquidity effect and the interest elasticity of money demand. The thought experiment that reveals a liquidity effect on interest rate is an open market operation, i.e., an increase of the money supply. Instead, the slope of the money demand is a relationship between real money balances, or velocity, and interest rates. As explained, in this model, the "long-run" interest-elasticity of the money demand is $-\rho$. The liquidity effect of an increase in the (nominal) money supply, too, depends on ρ, among other parameters.

As done in the literature, see e.g., Christiano, Eichenbaum, and Evans (1999), we define as the "short-run money demand elasticity" the ratio of the impact effect on real balances relative to the impact effect on interest rates following a monetary shock. We argue that there is no "constant" short-run elasticity of the money demand in the model.[10] We emphasize that this is consistent with the unstable

[10] More precisely, while the long-run elasticity depends solely on the preference parameter ρ, the "short-run elasticity" also depends on the eigenvalues that determine the gradual adjustment of real balances, i.e., the preference parameters governing the speed of adjustment (ρ/γ) as well as those governing the money supply.

estimates of the interest-elasticity of money demand equations that are obtained using high-frequency data.

To fix ideas consider the case where the growth rate of the money supply follows an AR(1) process with parameter θ. From our previous analysis we have that the decrease on impact of aggregate real balances after a shock to the money supply is given by (26):

$$\frac{1}{m}\frac{dm}{d\hat{\mu}}\bigg|_{m=\bar{m},\ \hat{\mu}=0} = \frac{\kappa}{\bar{m}} = \frac{\alpha}{\bar{m}}\frac{\theta}{1 - \phi\theta},$$

where $\alpha < 0$ and $0 < \phi < 1$. Hence, real balances decrease after a money growth shock, the more so the more persistent is the shock. From our analysis of the impact on interest rates of a monetary shock, (36), we have that

$$\frac{1}{r}\frac{dr}{d\hat{\mu}}\bigg|_{m=\bar{m},\ \hat{\mu}=0,\ n=\bar{n}} = -\frac{1}{\rho}\left[\frac{(1 - \lambda\omega)}{\lambda\omega}\frac{\bar{n}}{\bar{m}\bar{\pi}^2}\left(\left(1 - \frac{\bar{m}}{\bar{c}}\right)\zeta - \frac{\bar{m}}{\bar{c}}\frac{\bar{\pi}^2\varphi_1}{\bar{n}^N}\right) + \frac{\kappa}{\lambda\omega\bar{m}}\right],$$

where ζ is given in (27) and the expression for φ_1 is given in online Appendix A.7 by equation (A11). Thus, we define the short-run elasticity of the money demand as the ratio:

$$\eta \equiv \frac{r}{m}\frac{dm}{dr} = \frac{\frac{1}{m}\frac{dm}{d\hat{\mu}}}{\frac{1}{r}\frac{dr}{d\hat{\mu}}}\Bigg|_{m=\bar{m},\ \hat{\mu}=0,\ n=\bar{n}}.$$

To sign this expression notice that $(1/m)(dm/d\mu) < 0$, so the sign of the elasticity η depends on whether there is a liquidity effect or not. If there is a liquidity effect then the elasticity is positive on impact, otherwise, it is negative. We consider two interesting special cases:

$$\eta \equiv \frac{r}{m}\frac{dm}{dr} = 0 \quad \text{if} \quad \theta = 0, \qquad \eta \equiv \frac{r}{m}\frac{dm}{dr} = -\rho \quad \text{if} \quad \lambda = 1.$$

In the case of a once-and-for-all increase in the money supply ($\theta = 0$), expected inflation is constant, and thus aggregate real balances remain constant ($\kappa = 0$, and $\hat{m}_t = 0$). Thus, the impulse response of a purely transitory shock on the growth rate of the money supply μ_t will display a short-run interest elasticity of the money demand equal to zero. By inspecting Figure 4 one can see that nominal interest rates and real balances move in the same direction on impact (and in subsequent horizons) since the shock is not persistent. As the shock becomes permanent, i.e., as $\theta \to 1$, numerical simulations show that the impact elasticity converges to $-\rho$. Instead, in the case where $\lambda = 1$, i.e., when markets are not segmented, the short-run and long-run elasticities are the same, since the standard interest elastic money demand equation holds at all frequencies.

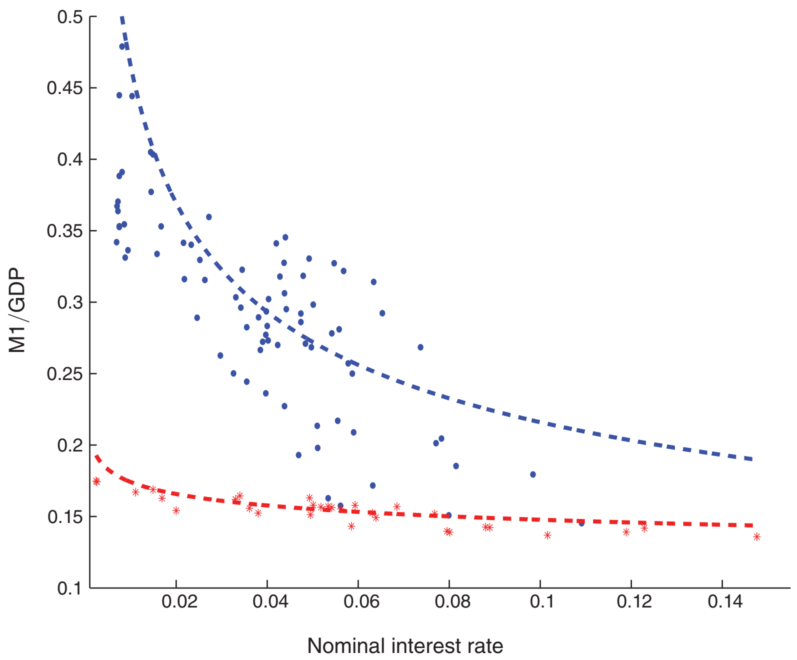

Figure 4. US Money Demand 1900–2010

Notes: Annual data denoted by dots refer to 1900–1980, stars refer to 1981–2010. The dashed lines plot a log-log money demands with a $1/3$ and $1/14$ interest elasticity respectively, which pass through the mean of both variables for each subperiod.

Source: Ireland (2009), updated to 2010

V. A Calibration of the Model on the US Data

In this section, we present a calibration of the model on the US data. The scatter plot in Figure 4 shows the annual US data on the money/income ratio and the nominal interest rate over the period 1900–2010 discussed, among others, by Lucas (2000) and Ireland (2009). The dots and stars denote, respectively, the pre- and post-1980 subsamples. This sample split follows Ireland (2009) who, as is apparent from the figure, estimates a significantly flatter (as well as lower) money demand schedule for the post 1980 sample. The goal is to use the model to simultaneously fit the apparent pattern of the money demand displayed by the lines in Figure 4 as well as the deviations from these schedules. In terms of the theory, the fitted lines correspond (roughly) to what we referred to as the long-run money demand, and the deviations correspond to the liquidity effects. Thus, the novelty is to use the dynamics of the model to account for the "errors" in the regression. Moreover, we will use the theory to quantify the downward bias on the estimation of the elasticity ρ using a standard regression analysis.

Table 2 presents regression estimates of the US money demand based on actual as well as on simulated data (described below). The estimates in the upper panel are obtained by regressing the (log) of inverse velocity on the (log) of the net nominal interest rate for three subsamples using US annual data over the 1900–2010 period.

Table 2—Actual versus Simulated Money Demand Regressions

	US data		
	1900–2010	1900–1980	1980–2010
$\hat{b}_1$	−0.23	−0.26	−0.06
R^2	0.28	0.56	0.69
DW	0.06	0.15	0.51
N	111	81	31
	Model simulations		
	$\lambda\omega = 0.5, \rho = 1/2$	$\lambda\omega = 0.60, \rho = 1/3$	$\lambda\omega = 0.65, \rho = 1/14$
$\hat{b}_1$	−0.22	−0.28	−0.07
R^2	0.22	0.54	0.67
DW	0.36	0.21	0.21
N	111 × 500	81 × 500	31 × 500

Notes: All regressions include a constant. The dependent variable is the inverse velocity: $M1/GDP$ in the data, m/c in the model. The R^2 is the squared errors of fit statistic, DW is the Durbin-Watson statistic. All coefficients are statistically different from 0 at the 1 percent level. The model statistics are the mean value obtained from 500 simulations.

Source: US annual data on M1 and interest rates are from Ireland (2009), updated to 2010.

We report the regression coefficient, the R^2 of the regression, and the DW statistic. The first column reports the full sample estimates, while the second and third present the estimates on the pre- and post-1980 subsamples. It is apparent that the regression coefficient $\hat{b}_1$ has become smaller (in absolute value). Its estimate goes from around $1/4$ in the first part of the sample to about $1/14$ in the more recent sample. In all regressions the residuals display a positive autocorrelation, as indicated by the low positive values of the Durbin-Watson statistic. The R^2 statistic appears high for such a simple model with only one explanatory variable and, consistently with the hypothesis of structural change put forward by Ireland (2009), the fit improves significantly in the subsamples.

The bottom panel of the table presents regression estimates based on artificial data produced by three different calibrations, one for each subsample. All calibrations assume that $U(c, m)$ is CRRA and $h(c, m)$ CES as in equation (12). We set the intertemporal substitution elasticity equal to $\gamma = 1/4$, as is common in the literature and within the range of estimates by Ogaki and Reinhart (1998) and others. The values of β and $\bar{\mu}$ are chosen so that the (unconditional mean) annual real interest rates and inflation rates are 2 percent (the values for our monthly model are $\beta = 0.9983$, and $\bar{\mu} = 1.0017$). Likewise, we impose a common statistical process for the money growth rate on all samples, given by the sum of two independent AR(1)'s to the monthly growth of M1 over the period of 1959–2009 estimated by maximum likelihood. We have chosen the sum of two AR(1)'s to capture the relative importance of high and low frequencies for money growth, a difference that is highlighted by our theory.[11] The estimates and their stability on the postwar subsamples are discussed in online Appendix B. One component is persistent, with a monthly

[11] In particular, the effect of each AR(1) can be understood by analyzing Figure 1 and Figure 2. Different measures of persistence are used in the literature, see, for example, the discussion in Marques (2004). Among them are

autocorrelation equal to 0.95, which corresponds to a half-life of about 1.1 years. The other component is transitory, with a monthly autocorrelation of 0.1, i.e., a half-life of a third of a month. The standard deviation of these innovations are 0.001 and 0.005, respectively, so that a large part of the year-to-year variation in money growth is explained by the more transitory component.

We calibrate two of the key parameters of the model, the interest rate elasticity ρ and the degree of segmentation $\lambda\omega$, to match the values of the regression coefficient $\hat{b}_1$ and the R^2 in each of the three subsamples. As mentioned, it is apparent in Figure 4 that the post 1980s period features a lower interest rate elasticity and higher velocity. In our model, this low frequency change can only be accommodated by shifts in ρ and $\mathcal{A}$, since segmentation does not affect the shape of the long run money demand. While ρ and $\mathcal{A}$ are preference parameters in our money in the utility function setup, other models can be used to produce a more structural explanation for these changes.[12] Likewise, the degree of segmentation $\lambda\omega$ is important to match the regression R^2. To see why, recall that in the case of no segmentation, $\lambda = 1$, there is no liquidity effect, so that the model-generated data obey the "long-run" money demand at each point in time and the regression fit is perfect (i.e., $R^2 = 1$). The two parameters have to be chosen jointly because, as explained below, lower values of $\lambda\omega$ imply a larger liquidity effect, which not only reduces the $R^{2,}$ but also increases the bias of $\hat{b}_1$ as an estimate of ρ. Such a calibration produces that the intratemporal elasticity of substitution is $\rho = 1/2,\ 1/3$, and $1/14$ for each subsample, and that the fraction of wealth owned by the traders is, respectively, $\lambda\omega = 0.5,\ 0.6$, and 0.65.[13]

In particular, to calibrate the model for each parameterization we produce 500 simulations of monthly data, which we aggregate to yearly, for real balances m and the nominal interest rate r. Each simulation has the same length as the three corresponding samples. For each simulation we run the same linear regression as in the data: $\log(m_t) = \hat{b}_0 + \hat{b}_1 \log(r_t) + \epsilon_t$. The three columns in the bottom panel of Table 2 report the mean of the estimated values of each statistic (the medians are virtually identical). The calibrated values are close, but not identical to the target values because we wanted to keep round values for the parameters ρ and $\lambda\omega$. As mentioned

the sum of the roots of an AR(p) process, and the largest root (or the dominant eigenvalue), which is the one we will use.

[12] Mechanisms that can give rise to these shifts are explored in Lucas and Nicolini (2012), Reynard (2004), and Alvarez and Lippi (2009), among others. In particular Lucas and Nicolini (2012) explore the effect of changes in banking regulations for demand deposits, and Alvarez and Lippi (2009) explore the effect of transaction technologies, such as availability and diffusion of ATM cards and terminals.

[13] Looking for direct empirical evidence on this parameter is not easy. To gauge the magnitude of it we used data from the US distribution of wealth to construct a relative wealth measure, ω_i, and a measure of households participation, λ_i, in bond (and/or equity) markets across the different quartiles of the financial wealth distribution. Our measure of household participation λ_i is the fraction of households in the quartile i with nonnegative holding on the relevant class (i.e., say bonds, equity directly held, etc.). The source data were taken from Guiso and Sodini (2012). We then measure the fraction of financial wealth owned by traders by computing $\lambda\omega = \sum_{i=1}^{4} \lambda_i \omega_i / 4$. The estimates vary depending on whether one considers bonds only, equity only and whether stockholding is measured as "direct" holding or includes "indirect holding" (say through mutual funds). Our model is about segmented bond markets, and hence participation refers to the holding and trading of the asset which are on the other side of an open market operation. Yet if households delegate their wealth management to financial intermediaries and have access to liquidity through them, one can consider that they participate through the holdings of their delegated portfolio. Overall, the estimates of $\lambda\omega$ range from a lower bound of 0.4 (when only direct equity holdings is considered) to an upper bound of 0.8 (when direct and indirect equity holdings are considered). We see the benchmark value used in the simulation as a reasonable starting benchmark to be used in the quantitative assessment.

above, the novel feature of model is its ability to explain both the long-run money demand, i.e., the fitted regression line, as well as its deviations, i.e., an R^2 less than one, even though it has only monetary shocks.

Table 2 also reports the mean value of the Durbin-Watson statistic, which we interpret as a measure of the persistence of the liquidity effect. The low value of the Durbin-Watson statistics points to a large positive autocorrelation of the residuals both in the data and in the model simulations. Interestingly, the model is able to produce highly autocorrelated deviations even though the persistent component of the exogenous money growth process has a much shorter half life.[14] This effect is due to the *endogenous* propagation caused by the liquidity effect as displayed in Figure 2, as well as in Table 1. Interestingly, the table shows that the fitted regression coefficient on simulated data produces a downward-biased estimator of the elasticity ρ, especially so in the whole sample and in the first subsample. This is due to the short-lived liquidity effect on our model, which produces changes in interest rates (the right-hand side variable) that are unrelated to velocity (the left- hand side variable in the regression). The liquidity effect ends up working as if it were measurement error in the interest rates, and thus produces the classical attenuation bias in the coefficient as an estimate of ρ. Instead, the persistent component gives rises to movement in velocity, which trace the long-run money demand. This is the same phenomenon as the difference between short- and long-run money demand elasticities studied in Section IVB. On comparing the two subsamples, note that the fit to the post 1980 US data has a higher R^2, which for the calibration requires a slightly higher value of $\lambda\omega$, a feature that we find plausible. Finally, this higher $\lambda\omega$ implies a very small bias in the estimate of ρ.

We conclude this section with a remark. Our estimated process for the growth rate of money supply has a relatively small variance of the low frequency component, which we use as the sole exogenous forcing process to solve for equilibrium inflation, real balances, and interest rates. Given the small estimated permanent component in the growth rate of money supply our simulated model produces a too small low-frequency variability in velocity compared to the data—a feature that was also apparent in Hodrick, Kocherlakota, and Lucas (1991). Modeling structural shocks that shift the money demand will also improve the fit in this dimension.[15]

VI. Concluding Remarks

This paper presented a theoretical model that gives rise to a persistent liquidity effect, and characterized its relationship with a long-run interest-elastic money demand. One key feature of our model is that the presence and the persistence of the liquidity effect is determined by a simple combination of the intertemporal elasticity of substitution and of the intratemporal elasticity of substitution of cash versus

[14] The persistent root of the growth rate of the *monthly* money supply is 0.95, while the autocorrelation of the residuals for the model is of the order of 0.9 in *annual terms*.

[15] Proposition 5 on the approximate aggregation in our model implies that it has the same implications for velocity and inflation, except at zero interest rates, as the cash-credit version of the model Hodrick, Kocherlakota, and Lucas (1991). Hence, we share the feature that the model produces a small variability of velocity for a process for money supply that resemble the one in the US data.

consumption. It was shown that the log-run (i.e., low frequency) properties of money demand determine strength and persistence of the short-run liquidity effects. This provided a unified theory for the low- and high-frequency movements of interest rates. The simplicity of the logic of the argument for the presence of a liquidity effect can be seen by noticing that a once-and-for-all increase of money supply must necessarily imply a liquidity effect.

Our interest in monetary models that feature liquidity effects based on segmented asset markets is to provide a framework for studying policy questions on the effects of monetary shocks, such as those analyzed in e.g., Lahiri, Singh, and Végh (2007); Nakajima (2006); Lama and Medina Guzman (2007); Khan and Thomas (2007); King and Thomas (2008); Bilbiie (2008); Cúrdia and Woodford (2008); and Zervou (2013). Comparing with this literature, we have deliberately kept the model as simple as possible. The simplicity allowed us to give a sharp characterization of the theoretical results, such as the relationship between money and prices in the presence of segmentation, and the impact effect and persistence of the liquidity effect as a function of simple elasticities. The disadvantage of this simplicity is that this version of the model lacks some features that may be interesting for some policy questions. For instance, the model has exogenous endowment, flexible prices, and constant exogenous participation rates in the bond markets. In this regard, we view the result of the model as applying to the aggregate nominal demand. We think that it is feasible and interesting to extend the model in several dimensions: to have variable inputs and to introduce nominal rigidities, so that the model can be used to study the "real" effects of monetary shocks and whether the liquidity effects produced by segmented markets are substitutes or complements to the nominal rigidities.

Another interesting use of the model will be to study impact on the strength and persistence of the liquidity effects of changes to the relative wealth of asset market participants, as well as of changes in participation rates, i.e., variations in $\omega\ \lambda$. This requires careful measures of both segmentation as well as strength and persistence of liquidity effects either across time or across countries. Furthermore, we can study the effect of shocks to ω. This will allow to guide empirical research similar to the one conducted in the "slow moving capital literature" such as the one in Mitchell, Pedersen, and Pulvino (2007) or Duffie (2010) and to study the welfare implications of different monetary policies designed to offset the effect of these shocks.

REFERENCES

Alvarez, Fernando, and Andrew Atkeson. 1997. "Money and exchange rates in the Grossman-Weiss-Rotemberg model." *Journal of Monetary Economics* 40 (3): 619–40.

Alvarez, Fernando, Andrew Atkeson, and Patrick J. Kehoe. 2002. "Money, Interest Rates, and Exchange Rates with Endogenously Segmented Markets." *Journal of Political Economy* 110 (1): 73–112.

Alvarez, Fernando, and Francesco Lippi. 2009. "Financial Innovation and the Transactions Demand for Cash." *Econometrica* 77 (2): 363–402.

Alvarez, Fernando, and Francesco Lippi. 2014. "Persistent Liquidity Effects and Long-Run Money Demand: Dataset." *American Economic Journal: Macroeconomics.* http://dx.doi.org/10.1257/mac.6.2.71.

Alvarez, Fernando, Robert E. Lucas, and Warren E. Weber. 2001. "Interest Rates and Inflation." *American Economic Review* 91 (2): 219–25.

Andrés, Javier, J. David López-Salido, and Javier Vallés. 2002. "Intertemporal substitution and the liquidity effect in a sticky price model." *European Economic Review* 46 (8): 1399–1421.

Bilbiie, Florin O. 2008. "Limited asset markets participation, monetary policy and (inverted) aggregate demand logic." *Journal of Economic Theory* 140 (1): 162–96.

Chari, V. V., Patrick J. Kehoe, and Ellen R. McGrattan. 1998. *Can Sticky Price Models Generate Volatile and Persistent Real Exchange Rates?* Research Department Staff Report 223. Federal Reserve Bank of Minneapolis.

Christiano, Lawrence J, and Martin Eichenbaum. 1992. "Liquidity Effects and the Monetary Transmission Mechanism." *American Economic Review* 82 (2): 346–53.

Christiano, Lawrence J., Martin Eichenbaum, and Charles L. Evans. 1999. "Monetary policy shocks: What have we learned and to what end?" In *Handbook of Macroeconomics*, Vol. 1, edited by John B. Taylor and Michael Woodford, 65–148. Amsterdam: Elsevier.

Cochrane, John H. 1998. "What do the VARs mean? Measuring the output effects of monetary policy." *Journal of Monetary Economics* 41 (2): 277–300.

Cogley, Tim, Thomas J. Sargent, and Paolo Surico. 2011. "The Return of the Gibson Paradox." Unpublished.

Cúrdia, Vasco, and Michael Woodford. 2008. "Credit frictions and optimal monetary policy." Unpublished.

Duffie, Darrell. 2010. "Presidential Address: Asset price dynamics with slow-moving capital." *Journal of Finance* 65 (4): 1237–67.

Duffie, Darrell, and Bruno Strulovici. 2012. "Capital Mobility and Asset Pricing." *Econometrica* 80 (6): 2469–2509.

Edmond, Chris, and Pierre-Olivier Weill. 2011. "Aggregate Implications of Micro Asset Market Segmentation." University of Melbourne Department of Economics Working Paper 1117.

Fuerst, Timothy S. 1992. "Liquidity, loanable funds, and real activity." *Journal of Monetary Economics* 29 (1): 3–24.

Gali, Jordi. 2002. "New Perspectives on Monetary Policy, Inflation, and the Business Cycle." National Bureau of Economic Research (NBER) Working Paper 8767.

Grossman, Sanford, and Laurence Weiss. 1983. "A Transactions-Based Model of the Monetary Transmission Mechanism." *American Economic Review* 73 (5): 871–80.

Guiso, Luigi, and Paolo Sodini. 2012. "Household Finance: An Emerging Field." Centre for Economic Policy Research (CEPR) Discussion Paper 8934.

Hevia, Constantino. 2008. "Maximum Likelihood estimation of an autoregressive moving average model ARMA (p, q)." http://siteresources.worldbank.org/DEC/Resources/Hevia_ARMA_estimation.pdf.

Hodrick, Robert J., Narayana R. Kocherlakota, and Deborah Lucas. 1991. "The Variability of Velocity in Cash-in-Advance Models." *Journal of Political Economy* 99 (2): 358–84.

Ireland, Peter N. 2009. "On the Welfare Cost of Inflation and the Recent Behavior of Money Demand." *American Economic Review* 99 (3): 1040–52.

Khan, Aubhik, and Julia Thomas. 2007. "Inflation and Interest Rates with Endogenous Market Segmentation." Federal Reserve Bank of Philadelphia Working Paper 07-1.

King, Robert G., and Julia K. Thomas. 2008. "Breaking the new keynesian dichotomy: Asset market segmentation and the monetary transmission mechanism." Unpublished.

Lagos, Ricardo, and Randall Wright. 2005. "A Unified Framework for Monetary Theory and Policy Analysis." *Journal of Political Economy* 113 (3): 463–84.

Lahiri, Amartya, Rajesh Singh, and Carlos Végh. 2007. "Segmented asset markets and optimal exchange rate regimes." *Journal of International Economics* 72 (1): 1–21.

Lama, Ruy, and Juan Pablo Medina Guzman. 2007. "Optimal Monetary Policy in a Small Open Economy Under Segmented Asset Markets and Sticky Prices." International Monetary Fund (IMF) Working Paper 07/217.

Lucas, Robert E., Jr. 1990. "Liquidity and interest rates." *Journal of Economic Theory* 50 (2): 237–64.

Lucas, Robert E., Jr. 2000. "Inflation and Welfare." *Econometrica* 68 (2): 247–74.

Lucas, Robert E., Jr., and Juan Pablo Nicolini. 2012. "On the stability of the money demand." Unpublished.

Lucas, Robert E., Jr., and Nancy L. Stokey. 1987. "Money and Interest in a Cash-in-Advance Economy." *Econometrica* 55 (3): 491–513.

Marques, Carlos Robalo. 2004. "Inflation persistence: facts or artefacts?" European Central Bank (ECB) Working Paper 371.

Mitchell, Mark, Lasse Heje Pedersen, and Todd Pulvino. 2007. "Slow Moving Capital." *American Economic Review* 97 (2): 215–20.

Nakajima, Tomoyuki. 2006. "Monetary policy with sticky prices and segmented markets." *Economic Theory* 27 (1): 163–77.

Occhino, Filippo. 2004. "Modeling the Response of Money and Interest Rates to Monetary Policy Shocks: A Segmented Markets Approach." *Review of Economic Dynamics* 7 (1): 181–97.

Ogaki, Masao, and Carmen M. Reinhart. 1998. "Measuring Intertemporal Substitution: The Role of Durable Goods." *Journal of Political Economy* 106 (5): 1078–98.

Reynard, Samuel. 2004. "Financial market participation and the apparent instability of money demand." *Journal of Monetary Economics* 51 (6): 1297–1317.

Rotemberg, Julio J. 1984. "A Monetary Equilibrium Model with Transactions Costs." *Journal of Political Economy* 92 (1): 40–58.

Sargent, Thomas J., and Paolo Surico. 2011. "Two Illustrations of the Quantity Theory of Money: Breakdowns and Revivals." *American Economic Review* 101 (1): 109–28.

Williamson, Stephen D. 2008. "Monetary policy and distribution." *Journal of Monetary Economics* 55 (6): 1038–53.

Williamson, Stephen, and Randall Wright. 2010. "New Monetarist Economics: Models." In *Handbook of Monetary Economics*, Vol. 3, edited Benjamin M. Friedman and Michael Woodford, 25–96. Amsterdam: Elsevier.

Zervou, Anastasia S. 2013. "Financial market segmentation, stock market volatility and the role of monetary policy." *European Economic Review* 63 (1): 256–72.